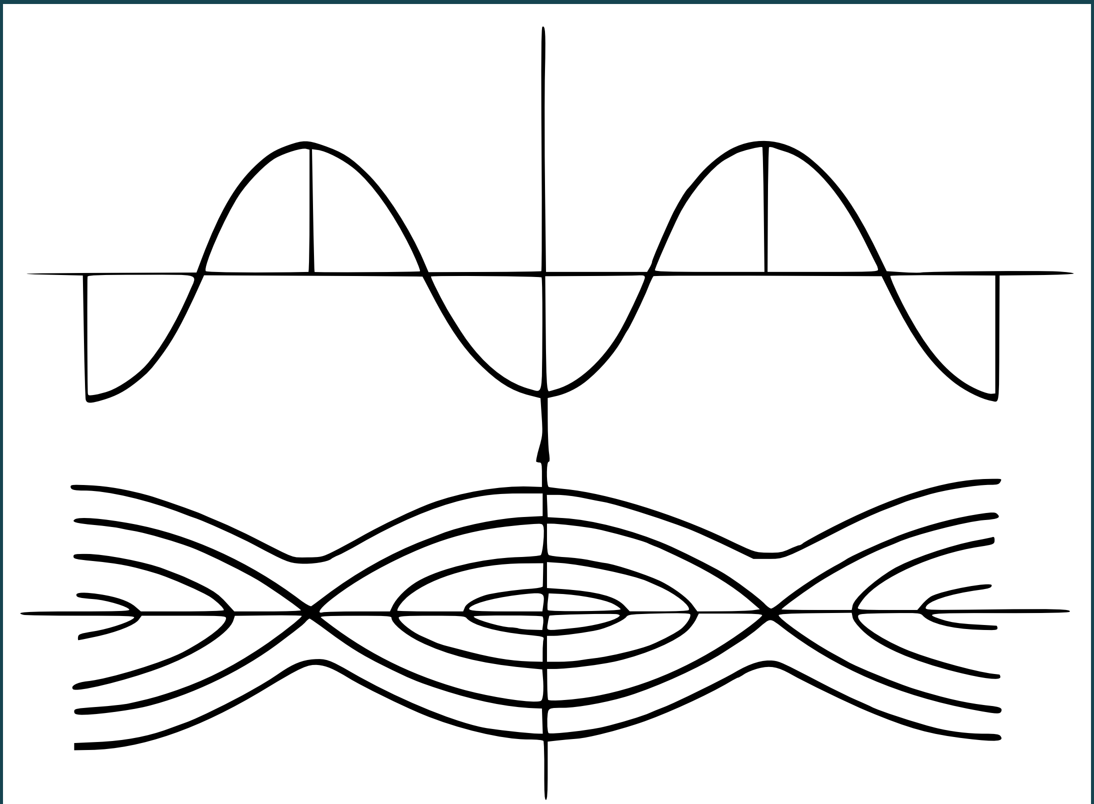
N.V. Butenin

Elements of the Theory of Non-Linear Oscillations



NONLINEAR OSCILLATIONS

Elements of the Theory of Nonlinear Oscillations is one of a special series of brief books covering selected topics in the pure and applied sciences.

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CONSULTING EDITOR

ELEMENTS
OF THE
THEORY OF NONLINEAR OSCILLATIONS

by N. V. Butenin

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Editor's Preface

This is an interesting, competent, useful book. The author presents, purely and simply, the main methods and facts of nonlinear mechanics. He does not merely catalog mathematical exercises; the book is a rich collection of examples which illuminate every side of the complex, fascinating subject of nonlinear mechanics. Prerequisites are minimal; this material should be easily accessible to anyone who has absorbed a reasonably modern first course in differential equations. It is quite suitable for a one-semester course on the senior level, and should be read by applied mathematicians, pure and applied physicists, aeronautical, electrical, and mechanical engineers, and anyone who wants to learn what the word "nonlinear" is all about.

Nonlinear mechanics has come to be regarded as a Russian specialty. This is due in large measure to the School of Mandel'shtam and Andronov and their followers. N. V. Butenin is a well-known member of this group. Indeed a book with so many relevant and up-to-date examples could not have been written without intimate contact with many years of research in nonlinear mechanics.

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NONLINEAR OSCILLATIONS

Introduction

Physical systems whose behavior is described by nonlinear differential equations are called nonlinear systems.

The theory of nonlinear vibrations is that part of the theory of vibrations which concerns vibrations in nonlinear systems.

The study of the behavior of nonlinear systems, that is, systems whose behavior is described by nonlinear differential equations, involves a deeper investigation of vibrational processes than does the study of linear systems. This follows from the fact that in the theory of linear vibrations a whole series of suppositions and conditions are introduced in setting up the differential equations, so that the resulting equations are linear. This method obviously helps in obtaining solutions of problems because the theory of linear differential equations is well-developed. However, at the same time the linearization of the equations means that a whole series of phenomena are neglected, such as self-excited oscillations, "tightening" and "gripping," and so on, which are characteristic of nonlinear systems, because these phenomena cannot be explained within the framework of linear theory. Moreover, there is a whole series of nonlinear problems which in principle cannot be linearized.

The theory of nonlinear vibrations gives a much deeper insight into vibrational systems and permits the study of phenomena which are not included in the linear theory of vibrations.

Obviously, nonlinear problems are considerably more diverse than linear ones and their study involves major difficulties.

Up to the present time we do not have a complete mathematical apparatus which would permit studying nonlinear phenomena with the same thoroughness with which the theory of linear vibrations is studied. However, a number of methods have been developed, both exact and approximate, which are adapted to the peculiarities of particular problems. These

methods make it possible to establish certain general principles, especially for the simplest systems, which are valid to a sufficient degree for whole classes of nonlinear problems.

The theory of nonlinear vibrations, as well as its application to the solution of various types of nonlinear problems, is treated in many monographs, and also in numerous technical papers, for instance, in the references [2], [3], [8], [10], [28], [31], [42].

The author of this book has sought to give the reader at least an introduction to the theory of nonlinear vibrations, to acquaint him with a few of the methods of studying nonlinear systems, and to apply these methods to the solution of concrete problems. Special attention has been paid to the study of self-excited oscillations.

It is proper that a book should exist which would serve as an introduction to the broad and diverse science of nonlinear oscillations. An acquaintance with the material developed in it should help the reader in a further, more detailed study of the theory of nonlinear oscillations and also in applying the techniques and devices developed here to the solution of other nonlinear problems.

I

Methods of Studying Nonlinear Autonomous Systems with One Degree of Freedom

1. The State Plane. State Diagrams of Linear Systems

The basic problems to be studied in nonlinear systems are the following: (1) to find the conditions of equilibrium of the system and investigate their stability; (2) to investigate the transition of the system from varying initial conditions to one or another stable state of motion; and (3) to investigate the dependence of the solution curves on the system parameters.

One technique for solving these problems is to represent the motion of the nonlinear system in a state plane or in a state space. Such a representation appeals to geometric intuition and simplifies the application of various methods of integrating the equations of motion. In several instances the method leads to results which would be difficult to obtain by other means.

The methods related to the state plane and state space have been developed and applied to the solution of many nonlinear problems in radio technology, mechanics, and automatic control by the Soviet scientists L. I. Mandel'shtam, N. D. Papalexsi, A. A. Andronov, and others. These methods have made possible the solution of many nonlinear problems and the explanation of specific properties of nonlinear systems.

We shall now look at the elements of the method of studying motions of a system with one degree of freedom by means of the state plane.

Many problems in the theory of oscillations lead to the study of a differential equation of the form

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right) = 0, \tag{1}$$

where $f(x,dx/dt)$ is a nonlinear function of x and dx/dt.

By introducing a new variable $y = dx/dt$, we can rewrite Equation (1) in the form of two equations of the first order:

$$\left.\begin{aligned} \frac{dy}{dt} &= -f(x,y), \\ \frac{dx}{dt} &= y. \end{aligned}\right\} \tag{2}$$

One may regard Equation (2) as a special case of a more general nonlinear system of the form

$$\left.\begin{aligned} \frac{dy}{dt} &= Q\,(x,y), \\ \frac{dx}{dt} &= P\,(x,y). \end{aligned}\right\} \tag{3}$$

Henceforth, our problem is to develop methods for studying the system (3). If the functions $P(x,y)$ and $Q(x,y)$ are independent of time, then the system (3) is said to be *autonomous*. In the converse case, the system is nonautonomous. In this chapter only'autonomous systems are considered.

Let us examine the plane with variables x and $y = dx/dt$, in which x and y are orthogonal Cartesian coordinates. To every state of a nonlinear system, defined by the coordinate x and the rate of change of this coordinate $y = dx/dt$, there corresponds a point in the xy-plane. Conversely, to every point of the xy-plane where the functions P and Q are defined there corresponds one and only one state of the system. Therefore the xy-plane is called the *state plane* (or the *phase plane.**) The point in the state plane which corresponds to a given state of the system is called the *representative point*.

When the state of the system changes to a new state there will be a new representative point. To a continuing change in the state of the system there corresponds a motion of the representative point in the state plane.

The curve described by the representative point in the state plane is called a (state) *trajectory*. The trajectory displays the dependence between the position and the velocity of the system under examination.

By a *complete trajectory* of a dynamical system we mean a curve which is composed of all representative points the system passes through in the course of time. The speed of the representative point in the state plane is the *state speed* v. The state speed is determined by the expression

* Editor's note: It is customary in physics to call $(x,\dot{x})$ the *phase coordinates* of a point and to call the $x\dot{x}$-plane the *phase plane*. Since $y \neq \dot{x}$ in Equation (3), it is preferable to talk of (x,y) as the *state variables* (or *state coordinates*) and call the xy-plane the *state plane*.

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{P^2(x,y) + Q^2(x,y)} \tag{4}$$

In the analysis of a nonlinear dynamical system there is special interest in the *equilibrium states,* that is those states where the velocity (dx/dt) and the acceleration $(dy/dt = d^2x/dt^2)$ are simultaneously equal to zero.

It is obvious that the equilibrium states of a dynamical system correspond to those points of the state plane where the state speed is zero, that is, where $dx/dt = 0$ and $dy/dt = 0$. According to equation (4) the state speed is zero at those points for which the following conditions are simultaneously satisfied:

$$\left.\begin{aligned} P(x,y) &= 0, \\ Q(x,y) &= 0. \end{aligned}\right\} \tag{5}$$

From Equation (3) it follows that the differential equation for the trajectories is

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{6}$$

In general, the solutions of the differential equation (6) cannot be expressed in such a way that y is an explicit function of x. But the solutions can always be expressed in the form

$$F(x,y) = \text{const.} \tag{6a}$$

If the value of the constant is held fixed, (6a) is the equation of a curve in the state plane. This curve may be connected or it may consist of several (disconnected) components. Such a curve is called an *integral curve* of (6). In general, each integral curve is composed of one or more trajectories.

Through every point of the state plane for which the state speed is not zero, that is, where the functions $P(x,y)$ and $Q(x,y)$ are not both zero, there will pass one and only one trajectory. This follows from the fact that at each such point the direction of the tangent to the trajectory is unique; this is one of the conditions which insures the uniqueness of a solution of a differential equation. (We are assuming that the other conditions included in this theorem for the functions P and Q have been satisfied.)

The state speed will be equal to zero at those points of the state plane for which conditions (5) are satisfied. These points will correspond to the equilibrium states, as has already been noted.

For the differential equation (6) such points are singular; at these points Cauchy's conditions for the uniqueness of the solution of a differential equation are not satisfied.

Thus, the singular points of the differential equation (6) can be inter-

preted physically as the equilibrium states of the dynamical system with which we are concerned.

We note that to a periodic motion of our system there corresponds a closed trajectory which does not pass through a singular point. This follows from the fact that the representative point, having begun its motion along the closed trajectory at an arbitrary point x and $y = dx/dt$, will return to the same point after one period. That is, the physical system will return to its previous state. After this, the trajectory described by the representative point will coincide with the previous trajectory. The length of time required to go around the closed trajectory, which is the period of the oscillation, will be finite because the state speed is bounded away from zero at all points of the trajectory.

If we are to know the behavior of trajectories in the whole state plane we must know, in particular, their behavior near the singular points. Therefore, it is necessary to examine first the nature of the states of equilibrium of a system.

We shall illustrate the concepts developed above by studying the simplest linear systems.

1. *The harmonic oscillator.* The equation for the harmonic oscillator, as is well-known, has the form

$$\ddot{x} + k^2x = 0. \tag{7}$$

Substituting $y = \dot{x}$, we get

$$\left.\begin{aligned} \frac{dy}{dt} &= -k^2x, \\ \frac{dx}{dt} &= y. \end{aligned}\right\} \tag{8}$$

Equation (6) in this case becomes

$$\frac{dy}{dx} = -\frac{k^2x}{y}. \tag{9}$$

This equation has a unique singular point which is the origin of the coordinate system, that is, $x = 0$ and $y = 0$.

Rewriting Equation (9) in the form

$$ydy + k^2xdx = 0$$

and performing the integration, we get

$$y^2 + k^2x^2 = c. \tag{10}$$

This is the equation of the integral curves of (9); it is a family of ellipses. Each of these ellipses (for a fixed value of the constant c) is a trajectory.

Thus, in this example of a simple linear oscillator the state plane is filled with nested ellipses, with the exception of the point $x = 0$, and $y = 0$. At this point the ellipse, which corresponds to the value $c = 0$, reduces to a point (Figure 1). Such a singular point, through which no trajectories pass and which is surrounded by closed trajectories, is called a center.

The speed of the representative point along the trajectory is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{y^2 + k^4x^2}.$$

As is obvious from this formula, the state speed is zero only when $x = 0$ and $y = 0$.

After having examined the state plane with the trajectories plotted on it (Figure 1), one can study the character of motion in the system.

It is clear that closed trajectories correspond to periodic motions. Consequently, we can say for the system under examination that a periodic motion will result for an arbitrary initial condition, except for $x = 0$ and

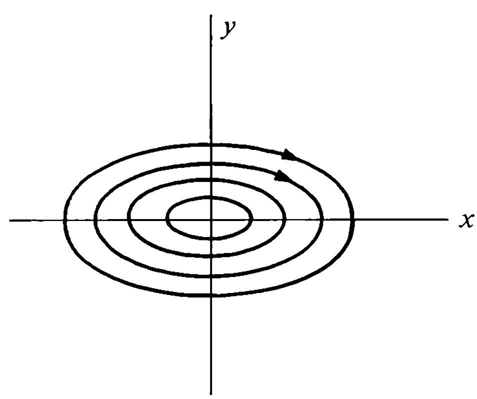


FIGURE 1

$y = 0$. The direction of motion along the trajectory is determined by Equation (8). If $x>0$ and $y>0$, then $dx/dt>0$ so that the representative point moves in the direction indicated in Figure 1.

We note that in this case every integral curve consists of a single trajectory.

2. *The harmonic oscillator with viscous damping.* The equation for motion in this case is

$$\ddot{x} + 2h\dot{x} + k^2x = 0.$$

Substituting $y = \dot{x}$, we get

$$\begin{aligned} \frac{dy}{dt} &= -2hy - k^2 x, \\ \frac{dx}{dt} &= y, \end{aligned} \tag{11}$$

and

$$\frac{dy}{dx} = \frac{-2hy - k^2 x}{y}. \tag{12}$$

The singular point in this case will be unique and is again at the origin ($x = 0, y = 0$).

To integrate **(12)** we make the change of variable $y = zx$, so that Equation **(12)** becomes

$$\frac{z\,dz}{z^2 + 2hz + k^2} = -\frac{dx}{x}.$$

Let us look first at the case when $h < k$ (the case of weak damping). Integration gives

$$x(z^2 + 2hz + k^2)^{\frac{1}{2}} = ce^{\frac{h}{k_1}\arctan\frac{z+h}{k_1}},$$

where $k_1 = \sqrt{k^2 - h^2}$.

Changing back to the old variables, we have

$$(y^2 + 2hxy + k^2x^2)^{\frac{1}{2}} = ce^{\frac{h}{k_1}\arctan\frac{y+hx}{k_1 x}}. \tag{13}$$

Equation (13) is the expression of the integral curves which fill the state plane. The state speed in this case is

$$v = \sqrt{(2hy + k^2x)^2 + y^2}. \tag{14}$$

From this formula it follows that the state speed is nowhere zero except at the point $x = 0, y = 0$ and that it converges to zero as this point is approached.

For more detailed analysis we introduce the new variables $u = k_1 x$, $v = y + hx$ and consider u and v as orthogonal coordinates. Then assuming that $k_1^2 = k^2 - h^2$ we get

$$y^2 + 2hxy + k^2x = (y + hx)^2 + k_1^2x^2 = u^2 + v^2$$

and, consequently,

$$u^2 + v^2 = c'e^{2\frac{h}{k_1}\arctan\frac{v}{u}}.$$

Changing to polar coordinates with

$$u = \rho \cos \varphi,\ v = \rho \sin \varphi,$$

we have

$$\rho^2 = c' e^{2\frac{h}{k_1}\varphi}$$

or

$$\rho = c e^{\frac{h}{k_1}\varphi}.$$

Therefore, the integral curves are spirals which wind around the origin (Figure 2). Every integral curve is composed of two complete trajectories: (1) the origin and (2) the remaining part of the integral curve. To understand this clearly, note that the singular point $u = 0$, $v = 0$, to which the representative point converges as it moves along a trajectory, cannot be reached in a finite amount of time. This is explained by the fact that the state speed [see Equation (14)] becomes arbitrarily small as the origin is approached. Therefore the representative point tends asymtotically to the singularity, but does not reach it in a finite time. This singular point is the limit of all trajectories with the exception of the one that corresponds to the initial condition $u = 0$, $v = 0$. A singular point of this type is called a *focus.*

We will now examine more closely the trajectories in the xy-plane.

It is obvious that a linear transformation $u = k_1 x$ and $v = y + hx$ cannot change the qualitative properties of the trajectories, that is, trajectories on the xy-plane will still be spirals for which the coordinate origin is a limit point. The character of the spirals can be explained in the following way: for small values of h/k_1 on the uv-plane the spirals will be similar to the circles $u^2 + v^2 = c'$. Under the above linear transformation, this circle is transformed into an ellipse $y^2 + 2hxy + k^2x^2 = \text{const}$. Conse-

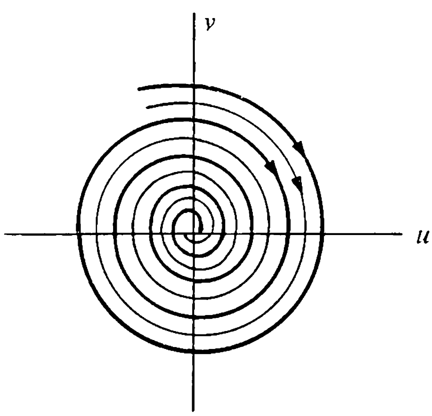


FIGURE 2

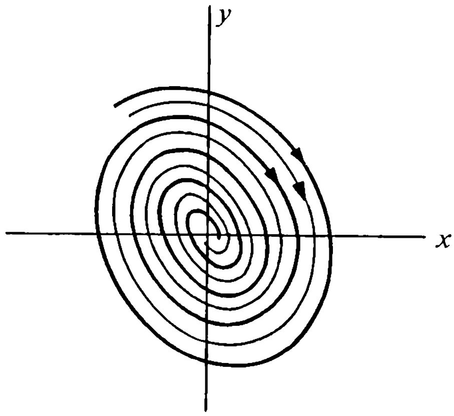


FIGURE 3

quently, for small h/k_1 the spirals on the xy-plane will approximate this ellipse (Figure 3).

The direction of motion of the representative point along the trajectory determined by Equations **(11)** is shown in Figure 3.

For an arbitrary initial condition (except for $x = 0, y = 0$), the representative point converges to the origin. Hence the equilibrium state of the system (which corresponds to the singular point at the origin) will be stable. Such a singular point is called a *stable focus*.

By examining the behavior of the trajectory one may conclude that the motion of the dynamical system will be a damped wave process for every initial condition, except $x = 0, y = 0$.

Let us examine now the case when $h > k$ (the case of strong damping). In this case the equation of the integral curves will be

$$y^2 + 2hxy + k^2x^2 = c\left[\frac{\frac{y}{x} + h - \sqrt{h^2 - k^2}}{\frac{y}{x} + h + \sqrt{h^2 - k^2}}\right]. \tag{15}$$

By introducing the notation

$$q_1 = h - \sqrt{h^2 - k^2} > 0,$$
$$q_2 = h + \sqrt{h^2 - k^2} > 0,$$

we can reduce Equation (15) to a simpler form. The left part of (15) is transformed in the following way:

$$\begin{aligned} y^2 + 2hxy + k^2x &= (y + hx)^2 - (h^2 - k^2)x^2 \\ &= [y + x(h + \sqrt{h^2 - k^2})][y + x(h - \sqrt{h^2 - k^2})] \\ &= (y + q_2x)(y + q_1x); \end{aligned}$$

therefore, Equation (15) takes the form

$$(y + q_1x)^{q_1} = c'(y + q_2x)^{q_2}.$$

Using the linear transformation

$$v = y + q_1x, \ \ u = y + q_2x$$

and considering u and v as orthogonal coordinates, we get

$$v = c_1u^a, \tag{16}$$

where $a = q_2/q_1 > 1$.

Let us examine more closely the significance of Equation (16).

For $c_1 = 0$ the integral curve will be the u-axis; for $c_1 = \infty$ the integral curve will be the v-axis.

Since $dv/du = ac_1u^{a-1}$, it follows that $dv/du = 0$ for $u = 0$, and, consequently, all integral curves are tangent to the u-axis at the coordinate origin.

All curves are convex relative to the u-axis because

$$\frac{v''}{v} = \frac{a(a-1)}{u^2} > 0.$$

The character of the integral curves is shown in Figure 4.

Let us go back now to the xy-plane.

The equation of the u-axis, which is tangent to all integral curves, will be $v = y + q_1x = 0$, that is, $y = -q_1x$, and the equation of the v-axis ($u = 0$) will be $y = -q_2x$.

We will now find the equation of the line which intersects the integral curves where they have horizontal tangents. Since at these points $dy/dx = 0$, it follows that

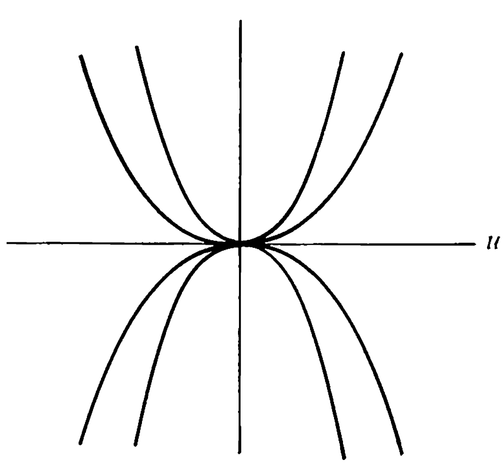


FIGURE 4

$$-2hy - k^2x = 0$$

and

$$y = -\frac{k^2}{2h}x.$$

It is easy to verify the fact that

$$\frac{q_1q_2}{q_1 + q_2} = \frac{k^2}{2h}.$$

A singular point of this type, which is an interior point of every integral curve,* is called a *node*. Since in this case the motion along every trajectory is directed towards the singular point, we have a *stable node*. A stable node corresponds to a stable equilibrium state and, consequently, the equation of the line will be

$$y = -\frac{q_1q_2}{q_1 + q_2}x,$$

where

$$\frac{q_1q_2}{q_1 + q_2} > q_1.$$

In Figure 5 there is a sketch of the integral curves on the xy-plane. The direction of motion is determined as it was earlier.

In the present example each integral curve is composed of three complete trajectories. Two of these trajectories correspond to the asymptotic

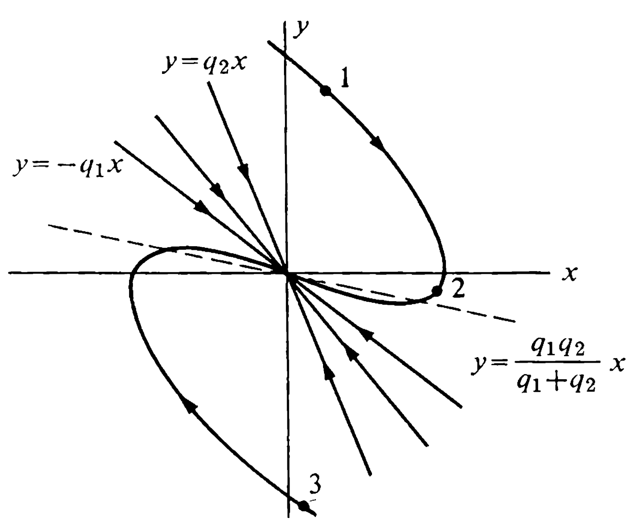


FIGURE 5

* The singular point in Figures 2 and 3 is a boundary point of every integral curve.

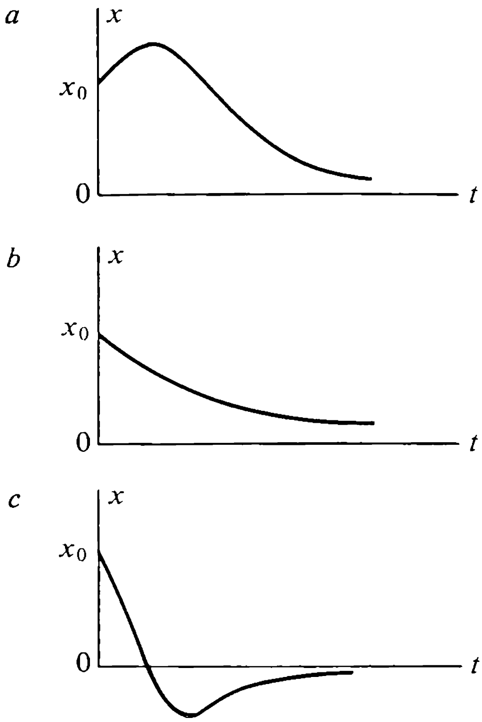


FIGURE 6

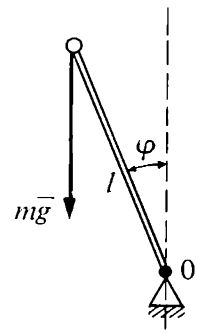


FIGURE 7

motion of the representative point as it approaches the origin, and the third is the singular point at the origin.

By examining the character of the trajectories (Figure 5) we can draw conclusions about the character of motion in the corresponding dynamical system for various initial conditions. For example, let $x_0 > 0$, $x_0 >$ be the point 1 in the state plane. Given these initial conditions, x at first increases up to the moment when its velocity becomes zero, and then decreases, that is, the system approaches an equilibrium state, though the speed at first is increasing in absolute value (the restoring force being greater than the damping force), and then, when the damping force becomes greater than the restoring force, the speed decreases asymptotically to zero. This case is shown in Figure 6*a*.

For $x_0 > 0$, $\dot{x}_0 < 0$ the character of motion is shown in Figure 6*b* and 6*c* where in the first case

$$|\dot{x}_0| < q_2 x_0 = x_0 \left(h + \sqrt{h^2 - k^2}\right),$$

and in the second case

$$|\dot{x}_0| > q_2 x_0 = x_0 \left(h + \sqrt{h^2 - k^2}\right).$$

3. By way of a third example let us look at the motion of a simple pendulum in the vicinity of unstable equilibrium.

The equation for the motion of a pendulum has the form (Figure 7)

$$ml^2\ddot{\varphi} = mgl \sin \varphi.$$

For small φ we get

$$\ddot{\varphi} - k^2\varphi = 0, \tag{17}$$

where $k^2 = g/l$.

Introducing the notation $\varphi = x$, and $\varphi = \dot{x} = y$, we have

$$\ddot{x} - k^2x = 0$$

and

$$\left.\begin{aligned} \frac{dy}{dt} &= k^2x, \\ \frac{dx}{dt} &= y. \end{aligned}\right\} \tag{18}$$

The system of Equations (18) has a unique singular point at the origin, that is at $x = 0, y = 0$.

Integrating the equation for the integral curves

$$\frac{dy}{dx} = \frac{k^2x}{y},$$

we get

$$y^2 - k^2x^2 = c. \tag{19}$$

Equation (19) defines a family of hyperbolas which fill the phase plane.

For $c = 0$ the integral curve degenerates into two lines $y = kx$ and $y = -kx$, which pass through the singular point. For $c \neq 0$ the integral curves do not pass through the singular point.

The lines $y = kx$ and $y = -kx$ are asymptotes for the integral curves for which $c \neq 0$.

The trajectories in the state plane are shown in Figure 8. The direction of motion of the representative point is determined, as usual, by one of the Equations (18).

From Figure 8 it follows that for arbitrary initial conditions, except for those which correspond to the asymptote $y = -kx$, the representative point will move away from the singular point, and thus the motion of the resulting system will be aperiodic. The state speed

$$v = \sqrt{k^4x^2 + y^2}$$

will be zero only at the singular point $x = 0, y = 0$. Even if the represent-

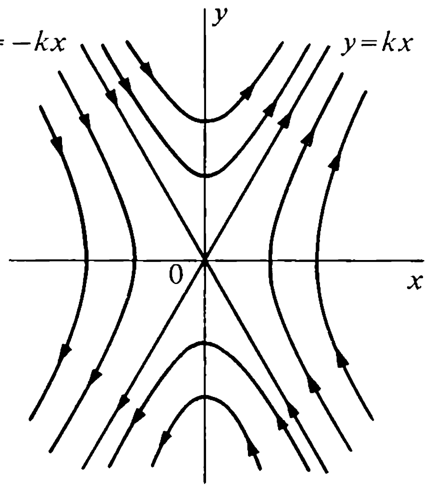


FIGURE 8

ative point is approaching the coordinate origin initially, it will eventually move arbitrarily far away from it.

Thus the equilibrium state which corresponds to the singular point is seen to be unstable. Such a singular point is called a *saddle point.*

The motion along the asymptote $y = -kx$ corresponds to the case where the representative point moves towards the origin with a speed approaching zero, but does not reach it in a finite interval of time. It is obvious that motion along this asymptote corresponds to a specially selected initial velocity, such that the initial kinetic energy is equal to the work which must be done by the force applied to the system in order that it arrive at the position of equilibrium. In practice, such a motion can never be realized because it is never possible to realize exact initial conditions. Therefore, a singular point of the saddle point variety is unstable for every "real" motion.

Each of the trajectories constructed above determine the character of all possible states and motions of the corresponding dynamical system. They are called the *state portraits* of the given system.

After having examined the state portraits of several of the simplest linear systems, one can make the following deductions:

a) Singular points and their character determine the character of the equilibrium state of the corresponding dynamical system.

b) The character and arrangement of the trajectories and the direction of the representative point along them makes it possible to visualize the resulting motion of the dynamical system.

In connection with this there arises the problem of finding the means and techniques of determining the singular points of nonlinear equations

and explaining their character. Also, there is the problem of finding methods for constructing the trajectories of these nonlinear equations, especially in cases where the equations of the integral curves cannot be obtained by elementary integrals.

Before going on to the solution of this problem, we will look at second-order linear systems of the general type and determine the character of the singular points which are possible in such systems. Such a study, as we shall see later, helps to solve the problem of the character of singular points for a large class of nonlinear systems.

2. Singular Points in Second-Order Linear Systems of the General Form

Now that we have studied the question of the possible types of singular points in linear systems, we shall look at the system of linear equations of the form

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned}\right\} \tag{20}$$

where a, b, c, and d are constant coefficients.

In what follows we shall suppose that $ad - bc \neq 0$.

In this case the point $x = 0, y = 0$ is the unique singular point of the system (20). In the converse case, the set of singular points is infinite and fills up the line

$$y = -\frac{a}{b}x = -\frac{c}{d}x.$$

As will be shown later, the latter case is not of practical interest for the solution of nonlinear problems because it reduces to the case of the zero root of the characteristic equation and precludes the possibility of comparing the characteristics of the resulting nonlinear system with its linear approximation.

The solution of the system (20) has the form

$$x = Ce^{st}, y = C\sigma e^{st}, \tag{21}$$

where C is an arbitrary constant.

It is necessary to select s and σ such that the solution (21) will make Equations (20) identities.

If we assume that $C \neq 0$, that is excluding the obvious solution $x = 0$, $y = 0$, which corresponds to the unique singular point of system (20), after the substitution of Equations (21) in (20) we have

$$a + \sigma b - s = 0,$$
$$c + \sigma d - \sigma s = 0,$$

from which

$$\left.\begin{aligned} \sigma &= \frac{s-a}{b}, \\ \sigma &= \frac{c}{s-d}. \end{aligned}\right\} \tag{22}$$

Eliminating σ from expression (22), we get an equation for s,

$$\frac{s-a}{b} = \frac{c}{s-d}$$

or

$$s^2 - (a+d)s + ad - cb = 0. \tag{23}$$

The expression (23) is called the *characteristic equation.*

Let the roots of this equation be s_1 and s_2. If we know these roots, s_1 and s_2, it is easy to determine that

$$\sigma_1 = \frac{s_1 - a}{b} = \frac{c}{s_1 - d}$$

and that

$$\sigma_2 = \frac{s_2 - a}{b} = \frac{c}{s_2 - d}. \tag{24}$$

The quantities σ_1 and σ_2 are called the coefficients of distribution and are the roots of the equation

$$b\sigma^2 + (a-d)\sigma - c = 0,$$

which is obtained by eliminating s from Equation (22).

The general solution of Equation (20) takes the form

$$\left.\begin{aligned} x &= C_1 e^{s_1 t} + C_2 e^{s_2 t}, \\ y &= \sigma_1 C_1 e^{s_1 t} + \sigma_2 C_2 e^{s_2 t}, \end{aligned}\right\} \tag{25}$$

where C_1 and C_2 are arbitrary constants of integration.

From the solution (25) it follows that the behavior of the integral curves and the type and character of the singular points are determined by the roots of the characteristic equation, which in turn depend on the values of the coefficients in system (20).

Let us examine the case when the roots of Equation (23) have real parts which are different from zero and are distinct. In this case the system of Equations (20) has, as was already shown, only one singular point

$$x = 0, y = 0. \tag{26}$$

We shall determine the character of this point. We will show that one can find a linear transformation

$$\left.\begin{aligned} \xi &= \alpha x + \beta y, \\ \eta &= \gamma x + \delta y \end{aligned}\right\} \tag{27}$$

such that Equation (20) is transformed into the form

$$\left.\begin{aligned} \frac{d\xi}{dt} &= A_1\xi, \\ \frac{d\eta}{dt} &= A_2\eta, \end{aligned}\right\} \tag{28}$$

where the coefficients A_1 and A_2 remain to be determined.

On the basis of (27) we have

$$\frac{d\xi}{dt} = \alpha\frac{dx}{dt} + \beta\frac{dy}{dt},$$

$$\frac{d\eta}{dt} = \gamma\frac{dx}{dt} + \delta\frac{dy}{dt}.$$

Substituting these expressions in (28) and making use of (20) and (27), we get

$$A_1(\alpha x + \beta y) = \alpha(ax + by) + \beta(cx + dy),$$

$$A_2(\gamma x + \delta y) = \gamma(ax + by) + \delta(cx + dy).$$

If we equate the coefficients of x and y on the right and left sides of these equations, we have:

$$\alpha A_1 = a\alpha + \beta c, \quad \gamma A_2 = \gamma a + \delta c,$$

$$\beta A_1 = b\alpha + \beta d, \quad \delta A_2 = \gamma b + \delta d$$

or

$$\alpha(a - A_1) + \beta c = 0, \quad \gamma(a - A_2) + \delta c = 0,$$

$$\alpha b + \beta(d - A_1) = 0, \quad \gamma b + \delta(d - A_2) = 0.$$

In order that α, β, γ, and δ are not all simultaneously equal to zero and

that the required transformation exists, it is necessary that the determinants of these systems of equations be equal to zero, that is that

$$\begin{vmatrix} a - A_1 & c \\ b & d - A_1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a - A_2 & c \\ b & d - A_2 \end{vmatrix} = 0,$$

which implies that

$$A_1^2 - (a + d) A_1 + ad - bc = 0$$

and that

$$A_2^2 - (a + d) A_2 + ad - bc = 0.$$

By comparing these equations with (23), we find

$$A_1 = s_1 \text{ and } A_2 = s_2,$$

where s_1 and s_2 are the roots of the characteristic equation.

Having determined A_1 and A_2, we arrive at the relations

$$\left.\begin{aligned} \frac{\alpha}{\beta} &= \frac{s_1 - d}{b} = \frac{c}{s_1 - a}, \\ \frac{\gamma}{\delta} &= \frac{s_2 - d}{b} = \frac{c}{s_2 - a}. \end{aligned}\right\} \tag{29}$$

We introduce the notation*

$$\kappa_2 = -\frac{\alpha}{\beta}, \quad \kappa_1 = -\frac{\gamma}{\delta}.$$

It is easy to prove that

$$\kappa_1 + \kappa_2 = -\frac{a - d}{b} \text{ and } \kappa_1\kappa_2 = -\frac{c}{b}$$

and, consequently, κ_1 and κ_2 are the roots of the equation which is used to determine the coefficients of distribution σ_1 and σ_2.

In this way, the system of Equations (20) reduces to the form

$$\left.\begin{aligned} \frac{d\xi}{dt} &= s_1\xi, \\ \frac{d\eta}{dt} &= s_2\eta. \end{aligned}\right\} \tag{30}$$

* We recall that the roots s_1 and s_2 are distinct and therefore by (29) $\alpha\delta - \beta\gamma \neq 0$, that is, the transformation to the canonical form is possible.

The linear nonsingular transformation (27) which we have found transforms the system (20) into the form (30), and considerably simplifies the analysis of the effect of the roots s_1 and s_2 on the character of the trajectories and the singular points of the system. In addition, it preserves all the qualitative properties of the state portrait for the resulting system.

The singular point $x = 0, y = 0$ on the xy-plane corresponds to the singular point $\xi = 0$, $\eta = 0$ on the $\xi\eta$-plane.

The behavior of the integral curves of the system (30) will depend on the character of the roots s_1 and s_2.

Let us examine the various cases.

1. Suppose s_1 and s_2 are real roots with the same sign.

The quotients α/β and γ/δ will be real numbers, and, consequently, ξ and η are real for real values of x and y. The equation for the integral curves will be

$$\frac{d\eta}{d\xi} = a\frac{\eta}{\xi}, \tag{31}$$

where $a = s_2/s_1 > 0$.

Integrating Equation (31), we get the family of integral curves given by

$$\eta = c\xi^a. \tag{32}$$

If $a > 1$, then the ξ-axis ($\eta = 0$) will be tangent to the integral curves; if $a < 1$, then the η-axis ($\xi = 0$) will be tangent to the integral curves. The arrangement of the integral curves is shown in Figure 9*a* and 9*b*.

When s_1 and s_2 are negative, the representative point moves towards the coordinate origin; a singular point of this type is called a *stable node*.

When s_1 and s_2 are positive, the representative point moves away from the coordinate origin and the singular point is called an *unstable node*.

Let us pass now to the xy-system of coordinates.

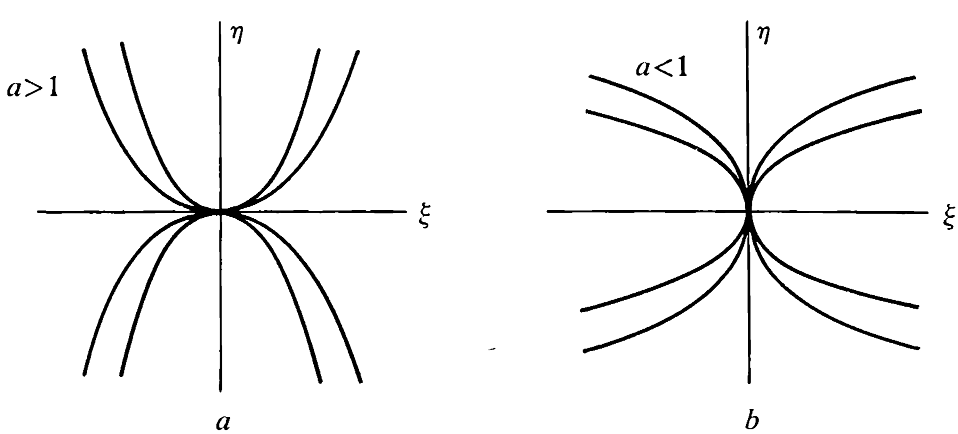


FIGURE 9

For $a > 1$, the tangent to the integral curves becomes

$$\eta = \gamma x + \delta y = 0.$$

that is,

$$y = -\frac{\gamma}{\delta} x = \lambda_1 x.^*$$

For $a < 1$, such a tangent will be

$$\xi = \alpha x + \beta y = 0,$$

that is

$$y = -\frac{\alpha}{\beta} x = \lambda_2 x.$$

These tangents are the integral curves for Equation (20).

Figure 10 shows the character of the integral curves when s_1 and s_2 are negative, and Figure 11 shows the case of positive s_1 and s_2.

2. Suppose that s_1 and s_2 are real with different signs.

The differential equation for the integral curves will be

$$\frac{d\eta}{d\xi} = -\left|\frac{s_2}{s_1}\right| \frac{\eta}{\xi} = -a\frac{\eta}{\xi},$$

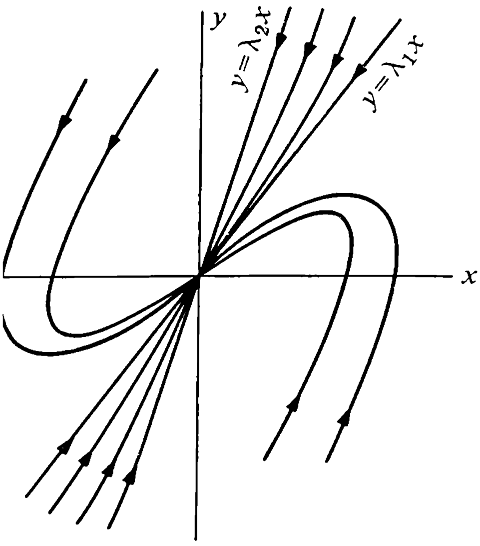


FIGURE 10

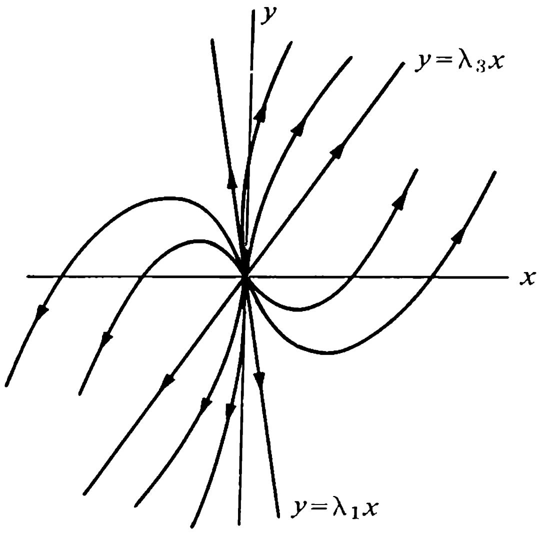


FIGURE 11

*This tangent will be tangent to all integral curves except for one, $x = 0, y = 0$, which is the singular point.

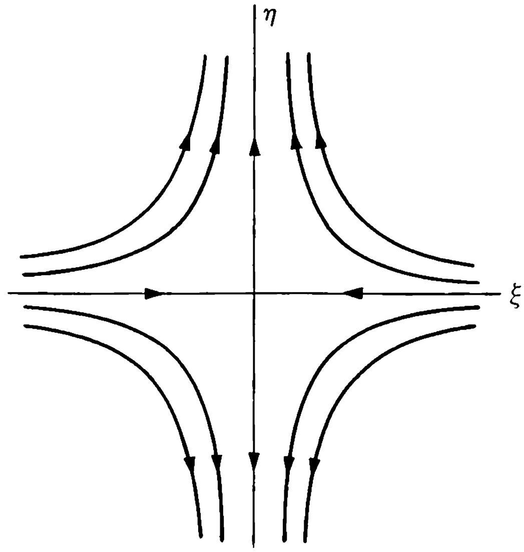


FIGURE 12

where

$$a = \left|\frac{s_1}{s_2}\right| > 0.$$

After integration we get the equation of the family of integral curves

$$\eta = \frac{c}{\xi^a}. \tag{33}$$

It is obvious that the axes $\xi = 0$ and $\eta = 0$ are both integral curves.

Equation (33) is an equation for curves of the hyperbolic type for which the ξ and η-axes are asymptotes. In Figure 12 a typical arrangement of the integral curves on the $\xi\eta$-plane is shown and also the direction of motion of the representative point for $s_1 < 0$ and $s_2 > 0$. The singular point $\xi = 0$, $\eta = 0$ is unstable and is called a *saddle point.*

On the xy-plane the asymptotes $\xi = 0$ and $\eta = 0$ are translated into the lines $y = \lambda_2 x$ and $y = \lambda_1 x$. The form of the integral curves on the xy-plane is shown in Figure 13.

3. Let the roots s_1 and s_2 be complex conjugates, that is

$$\left.\begin{aligned} s_1 &= a_1 + ib_1, \\ s_2 &= a_1 - ib_1, \end{aligned}\right\} \tag{34}$$

where a_1 and b_1 are real numbers.

Then for real variables x and y, the variables ξ and η will be complex conjugates.

We will show that one can find a linear transformation from the real variables x and y to new real variables u and v. Let

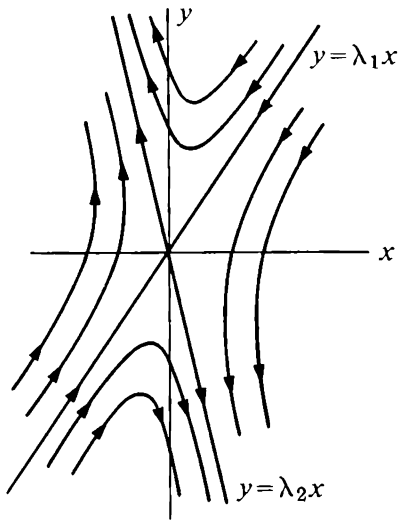


FIGURE 13

$$\left.\begin{aligned} \xi &= u + iv, \\ \eta &= u - iv, \end{aligned}\right\} \tag{35}$$

where u and v are real.

On the basis of (30) we have

$$\frac{du}{dt} + i\frac{dv}{dt} = (a_1 + ib_1)(u + iv),$$

$$\frac{du}{dt} - i\frac{dv}{dt} = (a_1 - ib_1)(u - iv),$$

from which

$$\left.\begin{aligned} \frac{du}{dt} &= a_1 u - b_1 v, \\ \frac{dv}{dt} &= b_1 u + a_1 v. \end{aligned}\right\} \tag{36}$$

The differential equation for the integral curves on the uv-plane will be

$$\frac{dv}{du} = \frac{b_1 u + a_1 v}{a_1 u - b_1 v}. \tag{37}$$

The point $u = 0$, $v = 0$ will be singular. For a more convenient picture we transfer to polar coordinates setting

$$u = \rho \cos \psi,$$

$$v = \rho \sin \psi.$$

Then

$$du = d\rho \cos \psi - \rho \sin \psi \, d\psi$$

and

$$dv = d\rho \sin \psi + \rho \cos \psi \, d\psi.$$

After substituting these expressions into Equation (37), we get

$$b_1 \rho d\rho = a_1 \rho^2 d\psi,$$

or, excluding from consideration the point $\rho = 0$, we have

$$\frac{d\rho}{\rho} = \frac{a_1}{b_1} d\psi.$$

The equation of the family of integral curves is

$$\rho = c e^{\frac{a_1}{b_1}\psi}. \tag{38}$$

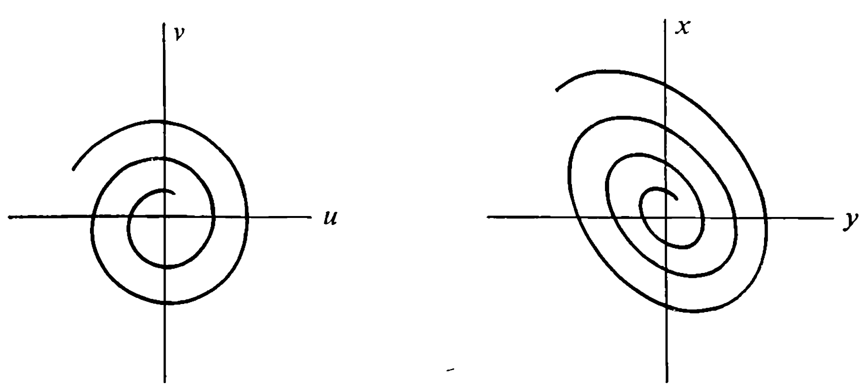
Therefore, the integral curves on the uv-plane are logarithmic spirals asymptotic to the point $u = 0$, $v = 0$. Such a singular point is called a *focus*. The form of the integral curves is shown in Figure 14*a*.

We will now find the direction of motion of the representative point. To do this we multiply the first equation of (36) by u and the second by v and combine to get

$$u\frac{du}{dt} + v\frac{dv}{dt} = a_1(u^2 + v^2).$$

Since $p^2 = u^2 + v^2$, finally we have that

$$\frac{d\rho}{dt} = a_1 \rho. \tag{39}$$



a *b*

FIGURE 14

Consequently, if $a_1 > 0 (a_1 = Re[s])$, then $\rho \to \infty$ and the representative point moves away from the origin. The singular point in this case is called an *unstable focus*.

If $a_1 < 0$, then $\rho \to 0$, that is, the representative point approaches the origin. The singular point will be a *stable focus*.

On the xy-plane we will also have spirals (Figure 14*b*).

Therefore, for the system of equations of the form

$$\frac{dx}{dt} = ax + by,$$

$$\frac{dy}{dt} = cx + dy,$$

with the conditions that $a + d \neq 0$, $ad - bc \neq 0$, and $(a + d)^2 \neq 4(ad - bc)$ the following five types of equilibrium states can exist:

1) for s_1 and s_2 real and negative, a stable node;
2) for s_1 and s_2 real and positive, an unstable node;
3) for s_1 and s_2 real and with different signs, a saddle point;
4) for s_1 and s_2 complex conjugates with $Re[s] > 0$, an unstable focus;
5) for s_1 and s_2 complex conjugates with $Re[s] < 0$, a stable focus. All these cases can be conveniently displayed in a diagram.

Suppose that

$$p = -(a + d),$$

$$q = ad - cb,$$

then the characteristic equation has the form

$$s^2 + ps + q = 0.$$

Its roots will be

$$s_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \quad s_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Let us now look at the qp-plane (Figure 15). For that part of the plane where $q < 0$, the singular points will be saddle points and, consequently, the equilibrium state will be unstable. The part of the plane where $q > 0$ and $p > 0$ corresponds to stable nodes and stable foci, that is to stable equilibrium states. In the part of the plane where $q > 0$ and $p < 0$, the singular points will be unstable nodes or unstable foci, and the equilibrium states will be unstable.

The curve $\Delta = p^2 - 4q$ separate foci from nodes.

Now let $a_1 = 0$ so that Equation (37) has the form

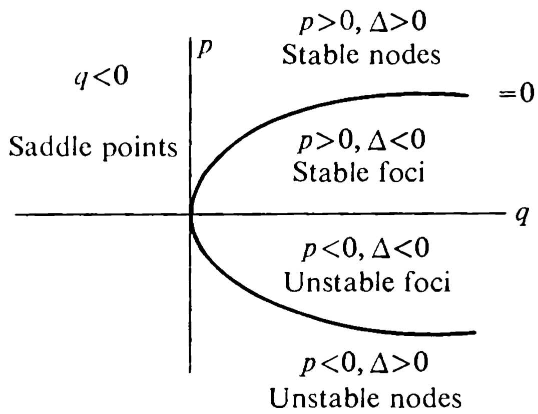


FIGURE 15

$$\frac{dv}{du} = -\frac{u}{v}$$

and, consequently,

$$u^2 + v^2 = \text{const.}$$

On the xy-plane this equation will have the form*

$$\frac{\alpha}{\beta}\frac{\gamma}{\delta}x^2 + y^2 + \left(\frac{\alpha}{\beta} + \frac{\gamma}{\delta}\right)xy = c',$$

or

$$by^2 - cx^2 + (a - d)xy = c.$$

Since

$$\frac{\alpha}{\beta} = -\lambda_2, \quad \frac{\gamma}{\delta} = -\lambda_1,$$

and

$$\lambda_1\lambda_2 = -\frac{c}{b}, \quad \lambda_1 + \lambda_2 = -\frac{a-d}{b},$$

this means that closed curves (ellipses)† surround the singular point. Such a point is called a *center*.

*The quantities u and v are expressed in terms of x and y by using formulas in Equations (27) and (35).

†We note that $a_1 = 0$ when $a + d = 0$, that is, when $a = -d$. Since in this case the roots s_1 and s_2 are imaginary conjugates, it follows that $ad - bc > 0$ and, consequently, that $-a^2 - bc > 0$, and this means that b and c have opposite signs, that is that the derived equation is the equation of an ellipse.

After the determination of the types and character of singular points in linear systems of the general form, it is possible to go on to the study of nonlinear systems.*

3. Lyapunov's Theorem on the Stability of an Equilibrium State

We suppose that the motion of a dynamical system is described by the system of equations

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x,y), \\ \frac{dy}{dt} &= Q(x,y), \end{aligned}\right\} \tag{40}$$

where x and y are the coordinates of the representative point on the xy-state plane, and where $P(x,y)$ and $Q(x,y)$ are nonlinear analytic functions of the variables x and y.

The equilibrium states of the dynamical system will correspond to the points at which the state speed

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

is equal to zero, that is, to the singular points of the differential equation satisfied by the integral curves

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{41}$$

The singular points of Equation (41) are found at

$$\left.\begin{aligned} P(x,y) &= 0, \\ Q(x,y) &= 0. \end{aligned}\right\} \tag{42}$$

Let the coordinates of a singular point be x_0, y_0. We shall determine the character and stability of this singular point, that is, we shall find the stability of the equilibrium state of the dynamical system.

As is well known [33], an equilibrium state in a dynamical system is said to be stable in the sense of Lyapunov if, given an arbitrary ε-neighborhood of the equilibrium state, one can always find a corresponding $\delta(\varepsilon)$-neighborhood such that a representative point located at the initial moment in the latter neighborhood never reaches the boundary of the ε-neighborhood.

---

*Note that the case $\Delta = (a + d)^2 - 4(ad - bc) = 0$ has not been considered.

Since we are interested in the stability of the singular point with coordinates x_0, y_0, we must study the character of motion of the representative point in a neighborhood of this equilibrium state. To this purpose we introduce new variables ξ and η, which characterize the deviation of the representative point from the equilibrium state

$$x = x_0 + \xi,$$

$$y = y_0 + \eta.$$

By expanding the functions $P(x,y)$ and $Q(x,y)$ in series in a neighborhood of the point x_0, y_0 in a power series in ξ and η, we have

$$P(x,y) = P(x_0,y_0) + P'_x(x_0,y_0)\,\xi + P'_y(x_0,y_0)\,\eta + P_1(\xi,\eta),$$

$$Q(x,y) = Q(x_0,y_0) + Q'_x(x_0,y_0)\,\xi + Q'_y(x_0,y_0)\,\eta + Q_1(\xi,\eta),$$

where $P(x_0,y_0) = 0$, and $Q(x_0,y_0) = 0$ since x_0 and y_0 are the coordinates of the singular point, and where $P_1(\xi,\eta)$ and $Q_1(\xi,\eta)$ are analytic functions in which the variables ξ and η occur with at least the second power.

Equations (40) then take the form

$$\left.\begin{aligned} \frac{d\xi}{dt} &= a\xi + b\eta + P_1(\xi,\eta), \\ \frac{d\eta}{dt} &= c\xi + d\eta + Q_1(\xi,\eta), \end{aligned}\right\} \tag{43}$$

where

$$a = P'_x(x_0,y_0), \quad b = P'_y(x_0,y_0),$$

$$c = Q'_x(x_0,y_0), \quad d = Q'_y(x_0,y_0).$$

If we neglect in Equation (43) the functions $P_1(\xi,\eta)$ and $Q_1(\xi,\eta)$, we get a system of linear differential equations with constant coefficients

$$\left.\begin{aligned} \frac{d\xi}{dt} &= a\xi + b\eta, \\ \frac{d\eta}{dt} &= c\xi + d\eta, \end{aligned}\right\} \tag{44}$$

which are called the *equations of the first approximation*.

The characteristic equation of the system (44) has the form

$$\begin{vmatrix} a - s & b \\ c & d - s \end{vmatrix} = 0$$

or

$$s^2 - (a + d)s + ad - bc = 0.$$

In the case when both roots of this equation have real parts different from zero, Lyapunov proved that the equilibrium state of the system (43) will be stable if the real parts of the roots are negative; if, however, one or both of the real parts is positive, then the equilibrium state will be unstable. If the real parts of both roots of the characteristic equation are zero, or if one root is zero and the other negative, then the equations of the first approximation (44) do not give an answer to the question of the stability of the equilibrium state.

In practical applications, one is concerned as a rule, with simple singular points, that is, those for which both roots have real parts different from zero.

The character of the state trajectories near simple singular points is also determined by the equations of the first approximation (44), [33], [36].

Thus the equations of the first approximation (44) can be used to determine the character and stability of the equilibrium state of the nonlinear system (40). The character and stability of the equilibrium states of system (44) have been studied in sufficient detail in the previous section.

Let us go on now to the application of the results of constructing trajectories on the state plane.

4. Methods of Constructing Trajectories

In this section we shall develop methods which make it possible in several cases to construct either exactly or approximately the integral curves on the state plane for a dynamical system with one degree of freedom.

THE METHOD OF ISOCLINES

The method of isoclines [2] is a general graphical method for constructing integral curves.

Let the equations for the motion of a dynamical system be

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x,y), \\ \frac{dy}{dt} &= Q(x,y). \end{aligned}\right\}$$

The equation for the integral curves on the state plane has the form

$$\frac{dy}{dx} = \varphi(x,y), \tag{45}$$

where

$$\varphi(x,y) = \frac{Q(x,y)}{P(x,y)}.$$

The curve

$$\varphi(x,y) = c, \tag{46}$$

for a fixed c, is called an *isocline;* it is the locus of points for which the slopes of the integral curves are the same, namely c.

The method of isoclines consists of the following. One constructs on the state plane a family of isoclines, that is, the curves (46) for various values of c. Let these values of c be $c_1, c_2, c_3, \ldots$. We note that the points of intersection of the isoclines will be singular points of Equation (45) because at these points the values of dy/dx become indeterminate.

Now we choose an arbitrary point A on the isocline for which $c = c_1$, and pass through it a line with slope c_1. Then we pass a line with slope c_2 through the same point (Figure 16). We divide the distance between the points of intersection of these lines and the isocline corresponding to $c = c_2$ into two equal parts and denote the midpoint by B. The line segment AB is taken as an arc of the integral curve between these two isoclines. For a more exact construction of this part of the integral curve it is necessary to choose values of c_1 and c_2 which are closer to each other. The construction is then continued in a similar way, starting with the point B.

EXAMPLE. Let us examine a dissipative system where the qualitative features of the trajectories can be obtained from the study of the type and character of the singular points and the construction of the isoclines.

We will investigate the motion of a simple pendulum in the presence of a viscous friction.

The differential equation of the system will be

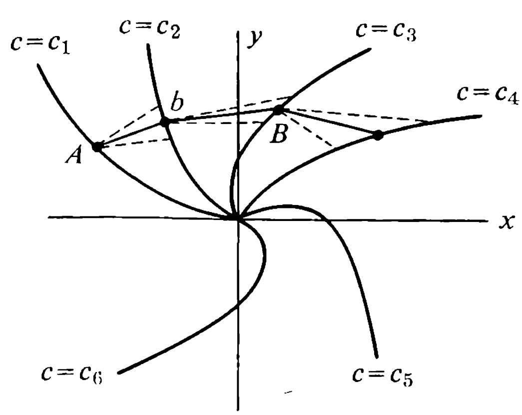


FIGURE 16

$$\ddot{x} + 2h\dot{x} + k^2 \sin x = 0 \qquad (h > 0,\ k^2 > 0).$$

Introducing the notation $\dot{x} = y$, we get two equations of the first order

$$\frac{dx}{dt} = y = P(x,y),$$

$$\frac{dy}{dt} = -2hy - k^2 \sin x = Q(x,y).$$

The differential equation for the integral curves has the form

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = \frac{-2hy - k^2 \sin x}{y}.$$

The singular points of Equation (46a) are determined by

$$-2hy - k^2 \sin x = 0,$$

$$y = 0.$$

All singular points are located on the x-axis; they have abscissas $x = 0, \pm\pi, \pm 2\pi, \ldots$.

We shall limit our study to three singular points $x_1 = 0$, $x_2 = \pi$, and $x_3 = -\pi$. Since x is the angular coordinate the curves on the state plane will be repeated after every interval of 2π. Since

$$P'_x = 0, P'_y = 1,$$

$$Q'_x = -k^2 \cos x, Q'_y = -2h,$$

for the first point $x_1 = 0, y_1 = 0$ the quantities p and q (see Section 3) will be

$$p = -(a + d) = 2h > 0,$$

$$q = ad - cb = k^2 > 0.$$

Thus, the equilibrium state $x_1 = 0, y_1 = 0$ is stable.

We shall determine the type of this equilibrium state. To this purpose we form the expression (see Sections 2 and 3)

$$\Delta = p^2 - 4q = 4(h^2 - k^2).$$

For $h > k$ the quantity $\Delta > 0$ and the equilibrium state will be a stable node; for $h < k$, $\Delta < 0$ and the equilibrium state is a stable focus.

For the equilibrium states $x_2 = \pi, y_2 = 0$ and $x_3 = -\pi, y_3 = 0$, the quantity $q = -k^2 < 0$, therefore, these equilibrium states are saddle points.

Tangents to the integral curves will be vertical at the x-axis and horizon-

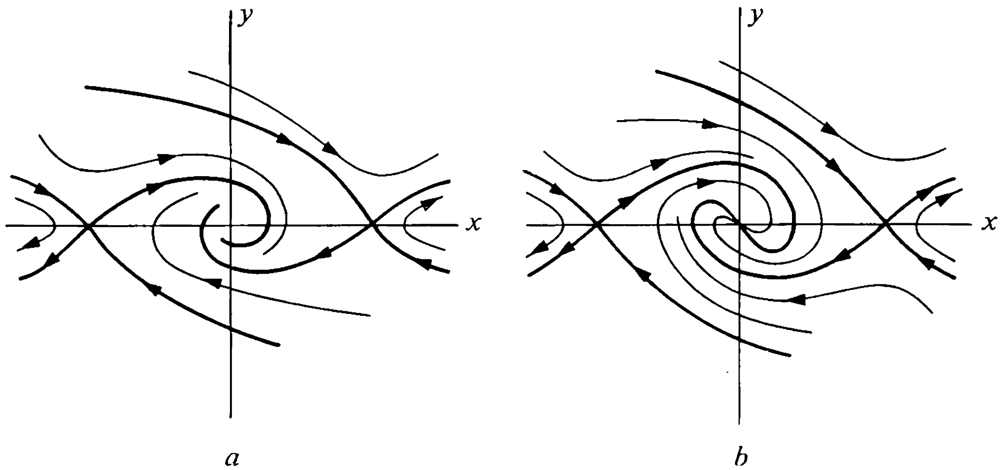


FIGURE 17

tal at the curve $y = -k^2/2h \sin x$. On the basis of these facts one can construct a qualitative graph of the arrangement of the trajectories. For $h < k$ such a graph is shown in Figure 17*a* and for $h > k$ in Figure 17*b*. From these drawings it follows that the system cannot have periodic motion.

It is interesting to note that the part of the state plane between $-\pi$ and π can be thought of as laid out on the surface of a cylinder.

Let us establish some more results for the construction of integral curves which are useful in several special cases.

CONSERVATIVE SYSTEMS

We shall look at the special case when the resulting dynamic system is conservative. The differential equation for such a system is

$$m\ddot{x} = f(x),$$

where $f(x)$ is an analytical function and m is a constant which for various dynamical systems has various physical meanings.

Introducing the notation $\dot{x} = y$ and $\varphi(x) = \dfrac{1}{m} f(x)$, we get

$$\left.\begin{aligned} \frac{dy}{dt} &= \varphi(x), \\ \frac{dx}{dt} &= y. \end{aligned}\right\} \qquad (47)$$

The equation for the integral curves will have the form

$$\frac{dy}{dx} = \frac{\varphi(x)}{y}.$$

This equation can be integrated. Since

$$y\,dy = \varphi(x)\,dx,$$

it follows that

$$\frac{y^2}{2} = h + \int \varphi(x)\,dx, \tag{48}$$

where h is the constant of integration.

The resulting expression is the equation for the integral curves. If this system is a mechanical system, the expression (48) is the law of conservation of mechanical energy since the function

$$\pi(x) = -\int \varphi(x)\,dx = -\frac{1}{m}\int f(x)\,dx$$

is the potential energy of the system divided by the mass m.

Therefore, Equation (48) can be rewritten in the form

$$\frac{y^2}{2} = h - \pi(x). \tag{49}$$

We note that for each value of the initial conditions ($t = t_0$, $x = x_0$, $y = y_0$) there corresponds one value of h, and for each value of h there corresponds an infinite set of values of x and y which satisfy Equation (49). The curve on the xy-state plane which is determined by Equation (49) is called a *curve of constant energy*. The branches of this curve are the trajectories.

The equilibrium states of the resulting dynamic system correspond to points where the state speed

$$v = \sqrt{y^2 + [\varphi(x)]^2}$$

is equal to zero.

According to (47) integral curves which have vertical tangents intersect the x-axis. From (49) it follows that the integral curves are symmetric relative to the x-axis. The points for which the integral curves have horizontal tangents are located on lines parallel to the y-axis which intersect the x-axis at the roots of the equation $\varphi(x) = 0$.

From Equation (49) it follows that for all x for which $h - \pi(x) < 0$ there will not be real branches of the integral curves.

Let us examine the auxiliary xz-plane on which we shall construct the function $z = \pi(x)$ (Figure 18*a*) and draw the line $z = h$. It is obvious that

real motions take place only for those x which are located to the left of the point of intersection of the curve $z = \pi(x)$ and the line $z = h$.

Since according to Equation (49)

$$\frac{y}{\sqrt{2}} = \pm\sqrt{h - \pi(x)}, \tag{50}$$

it follows that on the state plane $(x, y_1 = y/\sqrt{2})$ the integral curve will have the form shown in Figure 18*b*. The direction of motion of the representative point is determined by Equation (47) (for $y > 0$, x increases). For another value of h $(h = h_1)$ the integral curve is drawn as a dotted line.

Let us now look at the case when the function $z = \pi(x)$ has a minimum (that is, the potential energy has a minimum). This case is illustrated in Figure 19.

For $h = h_0$ the integral curve degenerates to a point where the state speed is zero (a singular point). For all $h > h_0$ the integral curves will be closed curves surrounding the singular point. Such a singular point is a center and is stable in the sense of Lyapunov.

Closed integral curves correspond to periodic motion in the resulting system. The period of this motion can be found from Equation (50). Since $\dot{x} = y$, it follows that

$$dt = \frac{dx}{y} = \frac{dx}{\sqrt{2[h - \pi(x)]}}$$

and, therefore, the period is equal to

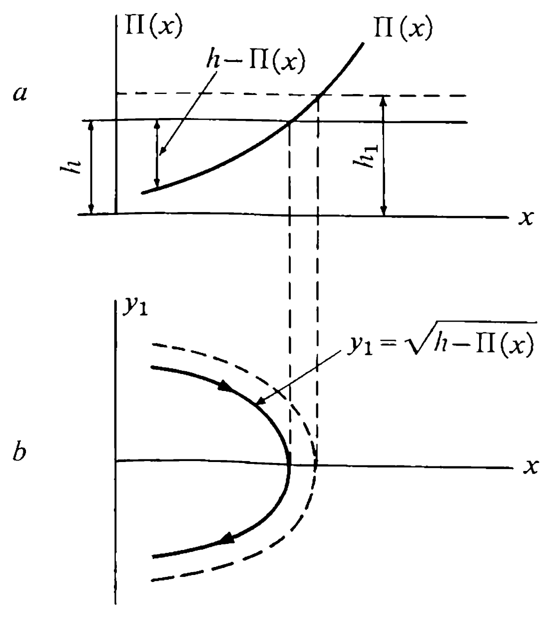


FIGURE 18

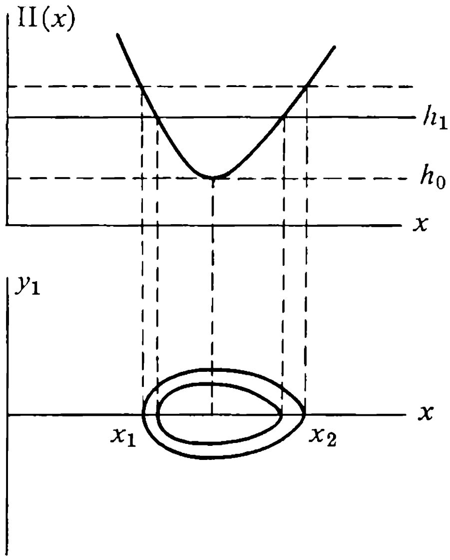


FIGURE 19

$$T = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{2\,[h - \pi(x)]}},$$

where x_1 and x_2 are the abscissas of the points of intersection of the integral curve with the x-axis.

Figure 20 shows the case when the potential energy has a maximum. When $h = h_0$ the integral curves consist of four trajectories which meet at one point. These trajectories are called *separatrices*. None of the other

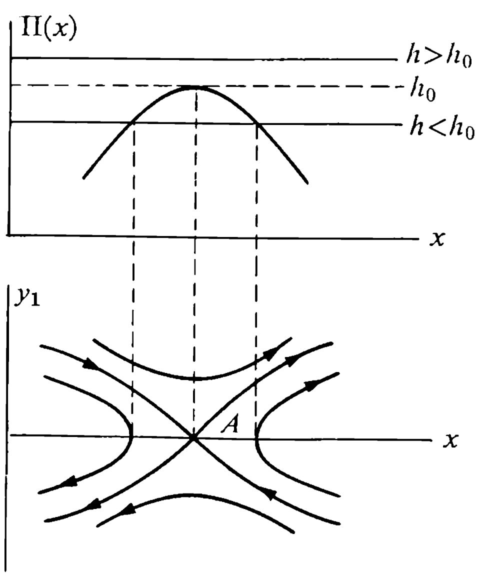


FIGURE 20

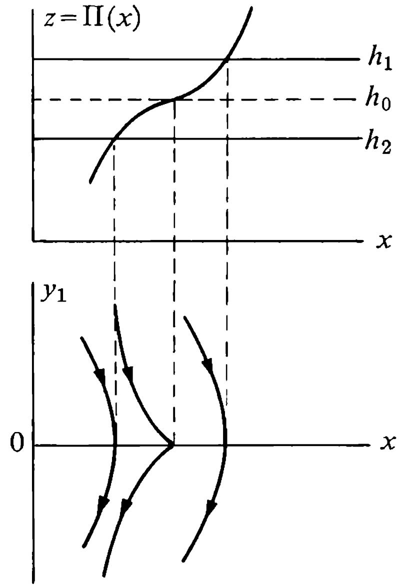


FIGURE 21

trajectories pass through the point A. For $h > h_0$ the trajectories are above and below A. For $h < h_0$ they are to the left and to the right of A. The character of the trajectories near the point A is that of a saddle point.

Finally, the case when the function $z = \pi(x)$ has a point of inflection is shown in Figure 21. The equilibrium state here is again unstable. (It is a singular point of a degenerate type, which may be thought of as resulting from the fusion of a saddle point (on the left of A) with a center (on the right of A).

The above examples constitute an elementary proof of the Lagrange-Dirichlet theorem: *If the potential energy has an isolated minimum at an equilibrium state, then the equilibrium state is stable.* And also of Lyapunov's theorem: *If the potential energy is not a mininum, at an equilibrium state, then the equilibrium state is unstable.*

Let us look at the possible motions on the whole state plane.

We shall suppose just as before that $z = \pi(x)$ is an analytic function for all values of x. We will graph this function on the xz-plane and deduce from this graph the character of the trajectories for any fixed value of h. The following cases are possible.

1. The curve $z = \pi(x)$ does not intersect the line $z = h$ anywhere. If the line $z = h$ lies below the curve $z = \pi(x)$, then there will be no motion anywhere on the state plane because in this case y is imaginary, that is, motion with this value of total energy is not possible.

If the line $z = h$ lies above the curve $z = \pi(x)$, then there will be two symmetrical branches of the integral curve (Figure 22).

For arbitrary initial conditions which correspond to points on the curve,

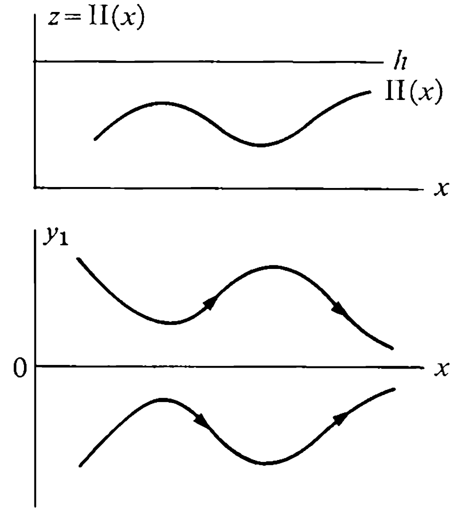


FIGURE 22

the representative point will approach either $x = \infty$ or $x = -\infty$. Such trajectories are called *divergent* (to infinity).

2. The line $z = h$ intersects the curve $z = \pi(x)$ but is nowhere tangent to it (Figure 23).

For all values of x for which $\pi(x) > h$ the trajectories do not exist. For all x for which $\pi(x) < h$ the trajectories can have two forms: closed arcs and an infinite arc. The closed arcs correspond to periodic motions; the infinite arc corresponds to divergent motions.

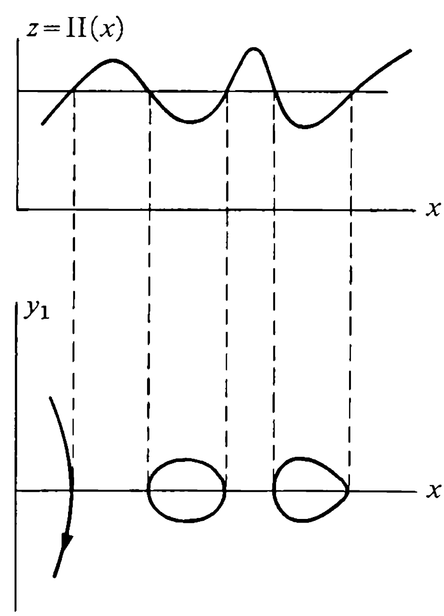


FIGURE 23.

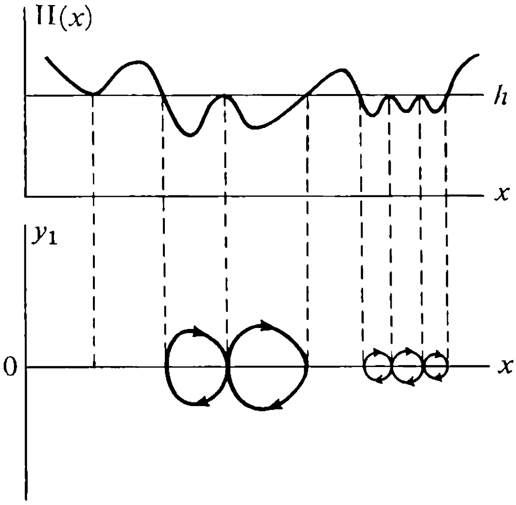


FIGURE 24

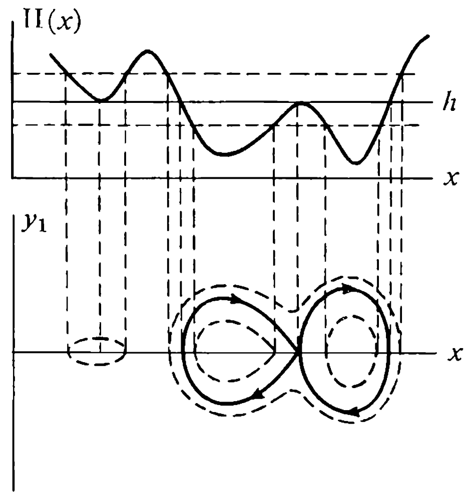


FIGURE 25

3. The line $z = h$ has several points of tangency with the curve $z = \pi(x)$. In this case the possible trajectories are of the following forms:
   a) isolated points, in the neighborhood of which there are no trajectories for the given h. When h is varied there will be either a closed trajectory (when h is increased), or there will be no real trajectories at all (when h is decreased);
   b) isolated finite arcs of trajectories (Figure 24); this is either a simple closed trajectory which corresponds to a periodic motion, or self-intersecting trajectories, which are called (closed) *separatrices*. The points of self-intersection of a closed separatrix are singular points of the saddle type and correspond to the points for which the line $z = h$ is tangent to the curve $z = \pi(x)$ at its maxima.

      Separatrices have great significance because they separate regions which contain trajectories of different types. In fact, by increasing h we get an integral curve which surrounds the whole separatrix; by decreasing h we get closed trajectories which are drawn in dotted lines in Figure 25.
   c) infinite arcs of trajectories. The form of the possible trajectories is shown in Figure 26. Here, there are trajectories which pass to infinity. These trajectories are also separatrices because with variations in h we get basically different trajectories.

Example 1. The pendulum. As is well known, the equation for the motion of a pendulum has the form

$$\ddot{\varphi} + k^2 \sin \varphi = 0,$$

where $k^2 = g/l$, φ is the angle of the pendulum with the vertical, l is the length of the pendulum, and g is the acceleration of gravity.

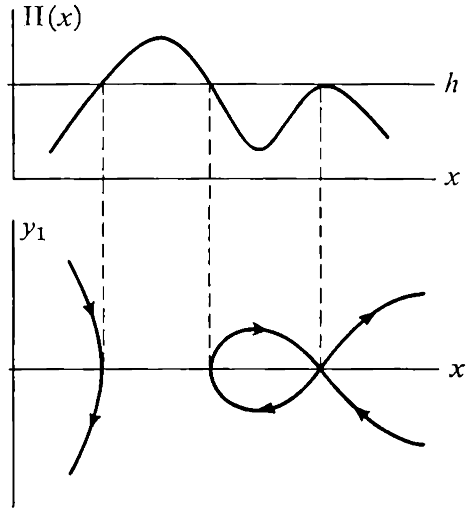


FIGURE 26

Introducing $\omega = \dot{\varphi}$, we have

$$\frac{d\omega}{dt} = -k^2 \sin \varphi, \qquad \frac{d\varphi}{dt} = \omega.$$

The function $z = \Pi(\varphi)$ in this case is

$$\Pi(\varphi) = \int^{\varphi} k^2 \sin \varphi d\varphi = -k^2 \cos \varphi$$

and, therefore, the equation for the integral curves has the form

$$\frac{\omega^2}{2} = h + k^2 \cos \varphi.$$

The function $z = \Pi(\varphi)$ (Figure 27) has isolated minima for $\varphi = 0$, $\pm 2\pi$, $\pm 4\pi$, ... These minima correspond to stable equilibrium states on the state plane, which are centers (Figure 27).

For the values $\varphi = \pm\pi$, $\pm 3\pi$, . . . the function $z = \Pi(\varphi)$ has isolated maxima; on the state plane they correspond to unstable equilibrium states, which are saddle points (Figure 27).

Figure 27 shows a sketch of the trajectories in the $\varphi\omega$-plane.

For $h < -k^2$ real motions of the pendulum do not exist.

For $k^2 > h > -k^2$ the trajectories are closed and surround singular points which are centers. These closed trajectories correspond to periodic oscillations of the pendulum.

For $h = k^2$ the trajectories are separatrices which intersect at saddle points.

If $h > k^2$, then the trajectories are divergent. But since the φ coordinate is periodic, the motion of the representative point along these trajectories

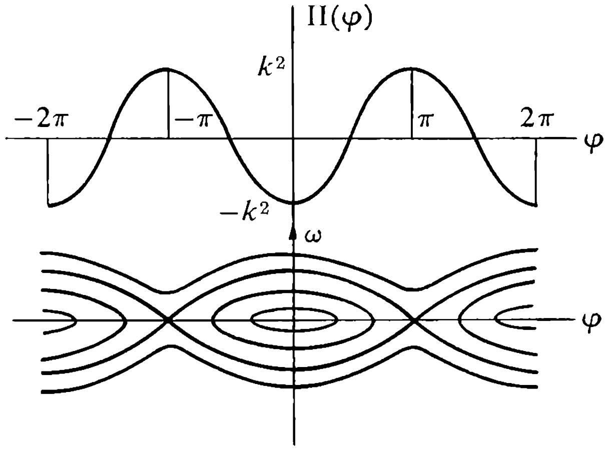


FIGURE 27

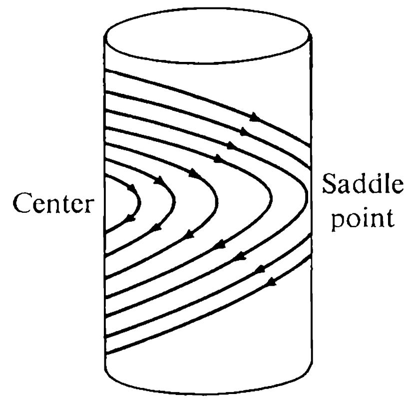


FIGURE 28

corresponds to a clockwise or counterclockwise rotation of the pendulum about its suspension point.

As was shown earlier, motion along the separatrix is possible only for specially selected initial conditions and, in practice, does not happen.

In conclusion, we note that in this case $\varphi\omega$-plane can be represented by the surface of a cylinder. On this cylinder there will be only two singular points, a center and a saddle point. On the cylinder the divergent trajectories will be closed (like a rubber band slipped on the cylinder). See Figure 28.

Example 2. The motion of a point mass along a parabola which is rotating about a vertical axis. This is a case where the equation for the trajectories cannot be put in the form of (49). However, the problem can still be solved by the method described above [2].

Suppose then that the point mass m moves freely along the parabola, the equation for which is $x^2 = 2pz$, and which is rotating around a vertical axis with a constant angular velocity ω (Figure 29).

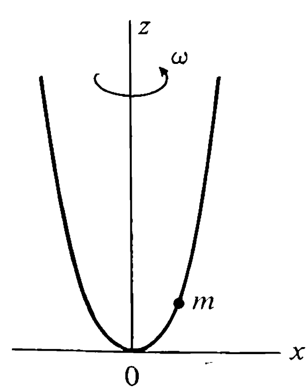


FIGURE 29

In order to derive the equation of motion we make use of Lagrange's equations of the second order. The system has one degree of freedom. As a generalized coordinate we shall choose x.

The kinetic energy of the system is

$$T = \frac{m}{2}(v_1^2 + v_2^2),$$

where $v_1 = \omega\,|x|$ is the rotational velocity of the point (perpendicular to the plane of the parabola); and where $v_2 = \sqrt{\dot{x}^2 + \dot{z}^2}$ is the relative velocity of the point (in the plane of the parabola).

Since $x^2 = 2pz$, it follows that $\dot{z} = x\dot{x}/p$.
Consequently,

$$T = \left(\frac{m}{2}\,1 + \frac{x^2}{p^2}\right)\dot{x}^2 + \frac{m}{2}\omega^2 x^2.$$

The expression for the potential energy is

$$\Pi = mgz = mg\frac{x^2}{2p}.$$

We form the Lagrange's function

$$L = T - \Pi = \frac{m}{2}\left(1 + \frac{x^2}{p^2}\right)\dot{x}^2 - \frac{m}{2}\lambda x^2,$$

where

$$\lambda = \frac{g}{p} - \omega^2.$$

We now form Lagrange's equation of the second order

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{51}$$

In the present case it will be

$$m\left(1 + \frac{x^2}{p^2}\right)\ddot{x} + m\frac{x\dot{x}^2}{p^2} + m\lambda x = 0,$$

from whence

$$\ddot{x} = -\frac{\left(\lambda + \frac{\dot{x}^2}{p^2}\right)x}{1 + \frac{x^2}{p^2}}.$$

Using the notation $\dot{x} = y$, we get

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{\left(\lambda + \frac{y^2}{p^2}\right)x}{1 + \frac{x^2}{p^2}}.$$

The equation for the integral curves is

$$\frac{dy}{dx} = -\frac{\left(\lambda + \frac{y^2}{p^2}\right)x}{\left(1 + \frac{x^2}{p^2}\right)y}. \tag{52}$$

Lagrange's Equation (51) after integration gives

$$\frac{\partial L}{\partial \dot{x}}\dot{x} - L = h,$$

and, therefore,

$$\frac{m}{2}\left[\left(1 + \frac{x^2}{p^2}\right)y^2 + \lambda x^2\right] = h.* \tag{53}$$

Equation (52) for $\lambda > 0$ and $\lambda < 0$ has one singular point at $x = 0$, $y = 0$. For $\lambda = 0$ there will be an infinite set of equilibrium states along the x-axis ($y = 0$). It is worth mentioning the fact that for $\lambda > 0$ the constant h must be positive, and for $\lambda < 0$ the constant h can be either positive or negative.

We rewrite Equation (53) in the form

$$y^2 = \frac{h'}{1 + \frac{x^2}{p^2}} - \lambda x^2, \tag{54}$$

where $h' = 2h/m$.

The values of x for which $y = 0$ are given by the equation

$$x^4 + p^2x^2 - \frac{h'}{\lambda} = 0,$$

and hence

$$x^2 = -\frac{p^2}{2} + \sqrt{\frac{p^4}{4} + \frac{h'}{\lambda}}. \tag{55}$$

From Equations (54) and (55) it follows that for $\lambda > 0$ the trajectories will be closed curves surrounding a stable singular point which is a center.

* This result is easily arrived at by direct integration of Equation (52).

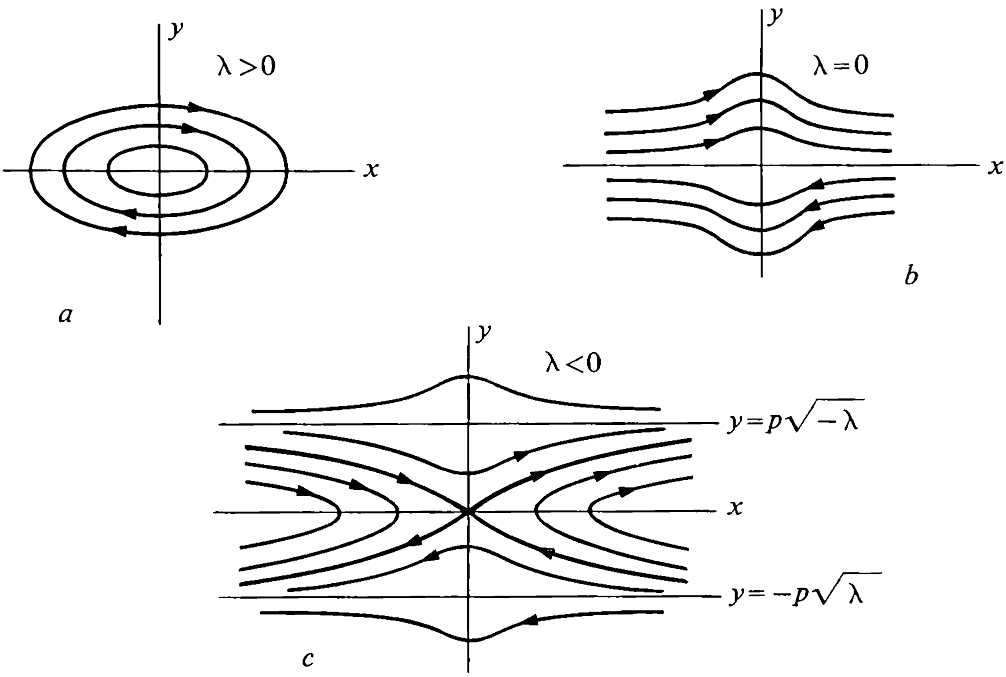


FIGURE 30

This follows from the fact that for a given h' the trajectories defined by (54) do not have infinite arcs. The character of the trajectories is shown in Figure 30*a*.

Therefore, for $\lambda > 0$ (that is, for $\omega^2 < g/p$) the mass point oscillates near the stable equilibrium state $x = 0, y = 0$.

For $\lambda = 0$ (that is, for $\omega^2 = g/p$) the trajectories do not cross the x-axis (Figure 30*b*). In this case the point will be either motionless at an arbitrary point of the parabola or will be moving uniformly along it on one side (this depends on the initial relative velocity).

Finally, if $\lambda < 0$ (that is, $\omega^2 > g/p$) the trajectories will intersect the x-axis for $h' > 0$ and will not intersect it for $h' < 0$.

The singular point will be a saddle point. The equations for the separatrix have the form

$$y = \sqrt{|\lambda|}\ \frac{x}{\sqrt{1 + \dfrac{x^2}{p^2}}}$$

and

$$y = -\sqrt{|\lambda|}\ \frac{x}{\sqrt{1 + \dfrac{x^2}{p^2}}}.$$

The character of the trajectories is shown in Figure 30*c*.

The lines $y = p\sqrt{-\lambda}$ and $y = -p\sqrt{-\lambda}$ are solutions of Equation (52) and correspond to the motion of the point along the parabola with a constant velocity.

For $|y| > p\sqrt{-\lambda}$ the motion of the point is similar to the case when $\lambda = 0$. For those y which satisfy the inequality

$$p\sqrt{-\lambda} > y > -p\sqrt{-\lambda},$$

the point either moves monotonically on one side, having a minimal velocity at the vertex of the parabola, or reverses its direction without reaching the vertex.

As is clear from what is stated above, the character of motion of the point depends essentially on the sign of the parameter λ, since as λ passes through 0 the character of the trajectories undergoes a qualitative change. The value of a parameter for which the character of the trajectories changes qualitatively is called a *bifurcation point*. In this case the bifurcation point occurs at $\lambda = 0$.

LIÉNARD'S METHOD [45]

Liénard's method is a graphical device for constructing the integral curves of a nonlinear equation of the form

$$\ddot{x} + \varphi_1(\dot{x}) + k^2 x = 0. \tag{56}$$

We introduce the change of variable $\tau = kt$, which gives

$$\dot{x} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} = \frac{dx}{d\tau} k$$

and

$$\ddot{x} = \frac{d^2x}{d\tau^2} k^2.$$

Then, Equation (56) takes the form

$$\frac{d^2x}{d\tau^2} + \varphi\left(\frac{dx}{d\tau}\right) + x = 0,$$

where

$$\varphi\left(\frac{dx}{d\tau}\right) = \frac{1}{k^2}\varphi_1\left(k\frac{dx}{d\tau}\right).$$

Letting $y = dx/d\tau$, we get

$$\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = -\varphi(y) - x.$$

The equation for the integral curves is

$$\frac{dy}{dx} = \frac{-\varphi(y) - x}{y}. \tag{57}$$

We now construct on the xy-state plane the curve

$$x = -\varphi(y).$$

We choose a point P in this plane with coordinates x and y (Figure 31). We draw a line from this point parallel to the x-axis to the point where it intersects the curve $x = -\varphi(y)$. We denote the point of intersection by the letter M. From M we drop a perpendicular MN to the x-axis. Since the length of the line segment MP is $x - [-\varphi(y)] = x + \varphi(y)$, the tangent of the angle of the line segment NP with the x-axis is

$$\tan\alpha = \frac{PL}{MP} = \frac{y}{x + \varphi(y)}.$$

According to Equation (57) the direction of the tangent to the integral curve passing through P will be perpendicular to NP.

Therefore the trajectories can be constructed in the following way: pass a line segment perpendicular to NP through the point P, choose a point P_1 on this segment and then determine the direction of the tangent to the integral curve in the same way, and so on. In this way we obtain a broken line which approximates the integral curve better and better as we make the line segments smaller and smaller.

Let us look at an example where it is possible to construct an exact integral curve using Liénard's construction.

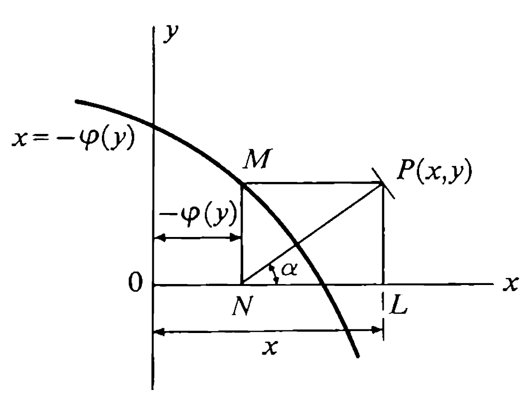


FIGURE 31

Example. The effect of Coulomb friction on a harmonic oscillator. The equation of motion of a harmonic oscillator with a constant friction force is

$$\ddot{x} + k^2 x = T_x,$$

where $T_x = -T$ for $\dot{x} > 0$ and $T_x = T$ for $\dot{x} < 0$ (T is positive).

If we let $\tau = kt$, we get

$$\frac{d^2x}{d\tau^2} + \varphi\left(\frac{dx}{d\tau}\right) + x = 0,$$

where

$$\varphi\left(\frac{dx}{d\tau}\right) = T_0 = \frac{T}{k^2} \qquad \text{for } \frac{dx}{d\tau} > 0$$

and

$$\varphi\left(\frac{dx}{d\tau}\right) = -T_0 = -\frac{T}{k^2} \qquad \text{for } \frac{dx}{d\tau} < 0.$$

We construct on the state plane the curve

$$x = -\varphi(y),$$

where

$$y = \frac{dx}{d\tau}.$$

It is obvious that for $y > 0$ the function $x = -\varphi(y)$ will be given by $x = -T_0$, and for $y < 0$, it will be given by $x = T_0$ (Figure 32).

By carrying out Liénard's construction for an arbitrary point on the state plane where $y > 0$, we easily convince ourselves that the integral curve

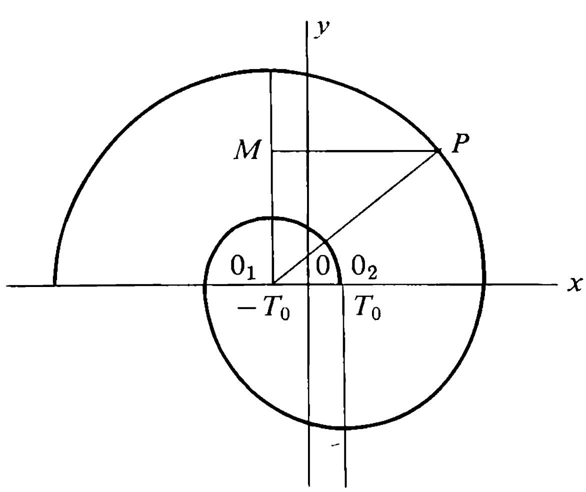


FIGURE 32

which passes through this point will be a circle with center at the point 0_1. For $y < 0$ the integral curves will be circles with center at the point 0_2. The form of the trajectories is shown in Figure 32. We note that the interval $0_1 0_2$ consists of "rest points," that is, there is no motion in this interval. At points in this interval the amount of elastic force in the spring is less than the force due to static friction.

DROBOV'S METHOD

This method applies to the equation

$$\ddot{x} + a\dot{x} + \frac{d}{dt}\{\Phi(x)\} + x + \Psi(x) = A, \tag{58}$$

where $\Psi(x)$ and $\Phi(x)$ are nonlinear functions, and a, A are constant coefficients. S. A. Drobov [22] suggested a graphical method for the construction of the integral curves in this case. We shall now explain the essence of this method.

Equation (58) can be rewritten in the form

$$\frac{d}{dt}\{\dot{x} + ax + \Phi(x)\} + x + \Psi(x) = A.$$

Introducing the notation

$$y = \dot{x} + ax + \Phi(x),$$

we have

$$\frac{dy}{dt} = A - x - \Psi(x),$$

$$\frac{dx}{dt} = y - ax - \Phi(x).$$

On the xy-plane the equation of the integral curves is

$$\frac{dy}{dx} = \frac{A - x - \Psi(x)}{y - ax - \Phi(x)}. \tag{59}$$

We construct on the xy-plane the curves

$$u = ax + \Phi(x),$$

$$v = \Psi(x),$$

and the line

$$\omega = A - x,$$

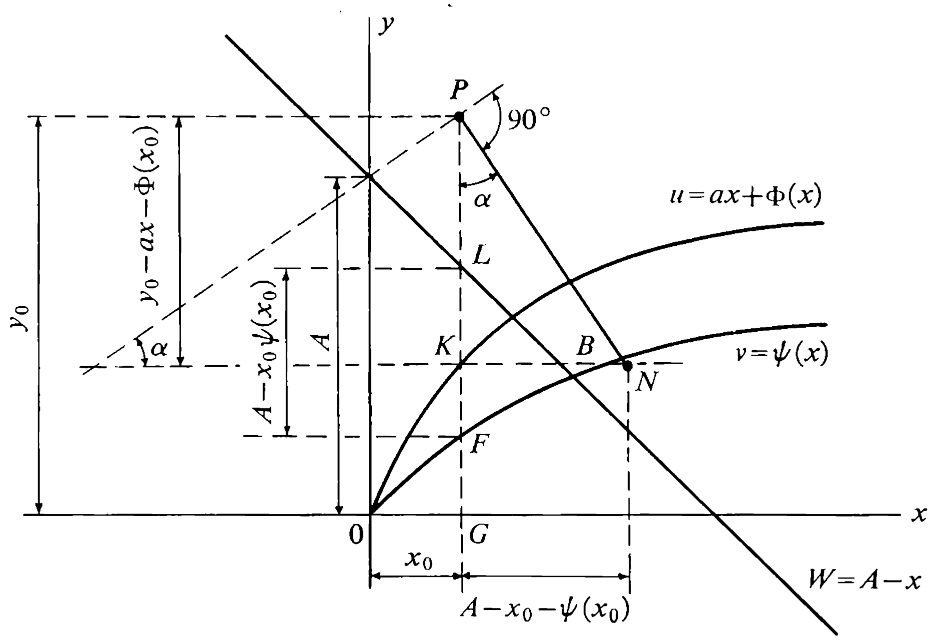


FIGURE 33

We regard u, v, and ω as variables corresponding to y.

We choose a point P on the xy-plane with coordinates x_0, y_0, through which we want to pass an integral curve (Figure 33). We drop a perpendicular from P to the x-axis so that we have

$$PK = PG - KG = y_0 - ax_0 - \Phi(x_0)$$

and

$$LF = LG - FG = A - x_0 - \Psi(x_0).$$

Then we pass a line parallel to the x-axis through the point K and pick out of this line the interval

$$KN = LF = A - x_0 - \Psi(x_0).$$

The interval KN is drawn in a positive direction along the x-axis if $A - x_0 - \Psi(x_0) > 0$ and in the opposite direction if $A - x_0 - \Psi(x_0) < 0$. By connecting the points P and N with a line segment we get (Figure 33)

$$\tan\alpha = \frac{A - x_0 - \Psi(x_0)}{y_0 - ax_0 - \Phi(x_0)}.$$

According to Equation (59) the tangent to the integral curve at P will be perpendicular to NP. We note that at an arbitrary point P_2 segment P_1G on the line, the tangent to the integral curve will be perpendicular to the line segment P_2N. (Figure 34)

In constructing the integral curves one should keep in mind the fact that

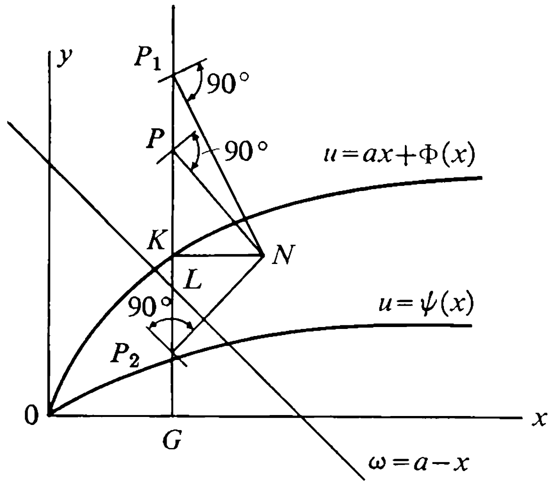


FIGURE 34

the curve $u = ax + \Phi(x)$ is a locus of points for which the integral curves have vertical tangents.

The locus of points for which the integral curves have horizontal tangents is the line

$$A - x - (x) = 0,$$

which is parallel to the y-axis and passes through the point B of intersection* of the curve $v = \Psi(x)$ and the line $\omega = A - x$.

The singular point(s) occur at the intersection of the curve $u = a\dot{x} + \Phi(x)$ and the line $A - x - \Psi(x) = 0$.

THE METHOD OF PELL [59]

Let us consider the equation

$$\ddot{x} + \varphi(\dot{x}) + f(x) = 0.$$

Letting $\dot{x} = y$, we get

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -\varphi(y) - f(x).$$

The equation for the integral curves is

$$\frac{dy}{dx} = \frac{-\varphi(y) - f(x)}{y}. \tag{60}$$

We construct on the xy-state plane the functions $\varphi = \varphi(y)$ and $f = f(x)$,

* In the general case there can be more than one such point of intersection.

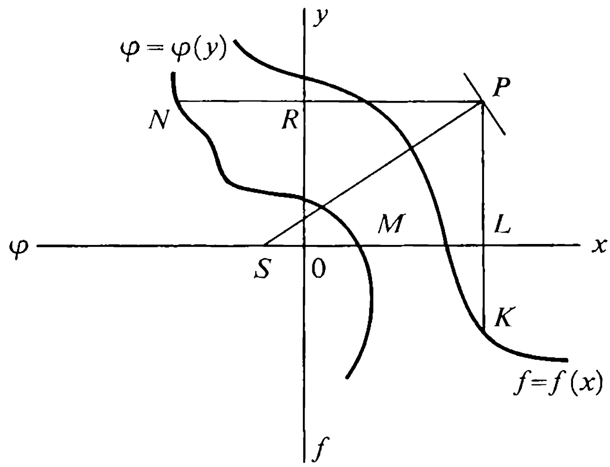


FIGURE 35

in such a way that φ is drawn along the negative x-axis and f is drawn along the negative y-axis (Figure 35).

To determine the direction of the tangent to the integral curve passing through an arbitrary point on the state plane we carry out the following construction: we pass a line parallel to the y-axis from the point P to the point K at the intersection of the line with the curve $f = f(x)$. Then the line segment LK is the value of the function $f(x)$ at P. We lay down this segment from the point L in the negative direction along the x-axis (if $f(x) > 0$) and get the segment LM. Then we pass a line parallel to the x-axis through the point P to the point N at the intersection of this line and the curve $\varphi = \varphi(y)$. The segment RN is the value of the function $\varphi(y)$ at P. Then we lay down this segment from the point M in a negative direction along the x-axis (if $\varphi(y) > 0$) and get the segment MS. It is obvious that

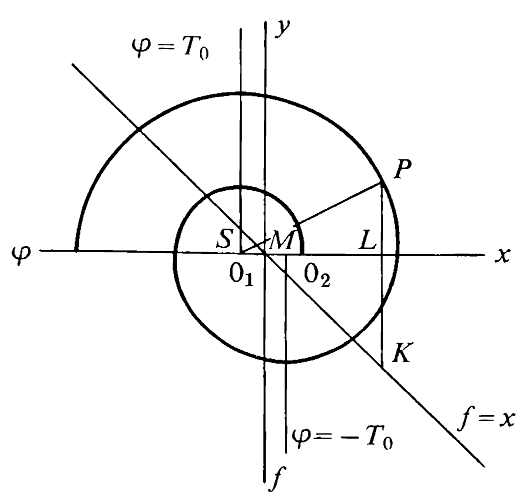


FIGURE 36

the tangent of the angle subtended by the segment SP with the x-axis is given by

$$\tan\alpha = \frac{PL}{LM + MS} = \frac{y}{f(x) + \varphi(y)}.$$

Therefore, according to Equation (60), the tangent to the integral curve will be perpendicular to the segment SP. Then we choose another point P_1 on this tangent close to the point P, repeat the above process of construction, and so on. The resulting broken line will approximate the integral curve better and better as the line segments are chosen smaller and smaller.

Figure 36 shows how to construct an integral curve by this method in the case of a harmonic oscillator with Coulomb friction (see Liénard's method).

THE METHOD OF "PIECING TOGETHER" REGIONS OF THE STATE PLANE [2]

We will study by this method the particular case of a harmonic oscillator with damping proportional to the square of the velocity.

The equation of motion for this system has the form

$$m\ddot{x} = -cx - b\dot{x}^2,$$

where $b > 0$ when $\dot{x} > 0$, and $b < 0$ when $\dot{x} < 0$.

Introducing the variable $\tau = t\sqrt{c/m}$, the equation takes the form

$$\frac{d^2x}{d\tau^2} + h\left(\frac{dx}{d\tau}\right)^2 + x = 0 \left(h = \frac{b}{m}\right),$$

where $h > 0$ when

$$\frac{dx}{d\tau} > 0$$

and $h < 0$ when

$$\frac{dx}{d\tau} < 0.$$

By letting $y = dx/d\tau$, we rewrite this equation in the form of two equations of the first order:

$$\left.\begin{aligned} \frac{dy}{d\tau} &= -x - hy^2, \\ \frac{dx}{d\tau} &= y. \end{aligned}\right\} \tag{61}$$

If we divide the first equation by the second we obtain the differential equation of the integral curves,

$$\frac{dy}{dx} = \frac{-x - hy^2}{y}. \tag{62}$$

The point $x = 0$ and $y = 0$ is the only singular point of this equation.

We consider first the case where $h > 0$ for *all* values of $dx/d\tau$. Then Equation (62) can be rewritten in the form

$$y\frac{dy}{dx} + hy^2 + x = 0$$

or

$$\frac{1}{2}\cdot\frac{dz}{dx} + hz + x = 0,$$

where $z = y^2$.

This is a linear nonhomogeneous differential equation of the first order. Its general solution is

$$z = ce^{-2hx} + \frac{1}{2h^2} - \frac{x}{h}.$$

Changing back to the variable y we get the equation of the integral curves

$$y^2 = \frac{1}{2h^2}(2ch^2e^{-2hx} + 1) - \frac{x}{h}. \tag{63}$$

This equation does not have real solutions for $c < -1/2h^2$.

For $c = -1/2h^2$ the equation has a single real solution corresponding to the point $x = 0$ and $y = 0$.

For $-1/2h^2 < c < 0$ the integral curves are closed.

For $c > 0$ the ends of the integral curves tend toward infinity.

The integral curve obtained from Equation (63) for $c = 0$, that is,

$$y^2 = \frac{1}{2h^2} - \frac{x}{h}, \tag{64}$$

is a separatrix and divides the region filled by closed trajectories from the region where the trajectories are not closed. The equilibrium state at $x = 0$ and $y = 0$ is a center.

The closed trajectories correspond to periodic motions: for all initial conditions lying inside the parabola (64) we have periodic motions. The direction of motion of the representative point along a trajectory is easily determined with the help of the second equation of (61). The picture of trajectories in the state plane is shown in Figure 37. It is worth mentioning

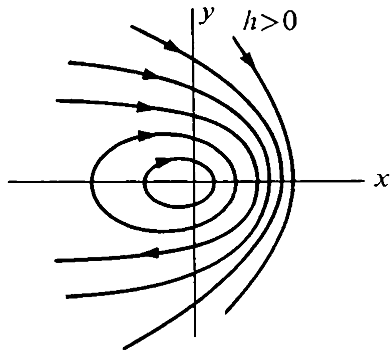


FIGURE 37

that the isocline for $dy/dx = 0$ is determined by the equation

$$y^2 = -x/h.$$

Consider now the case where $h < 0$ for *all* values of $dx/d\tau$; we note that Equation (62) does not change when h changes to $-h$ and x changes to $-x$. Consequently, in this case the picture of trajectories in the state plane remains the same as in the previous case, except that the direction of the x-axis is reversed (Figure 38).

If we return to the original problem ($h > 0$ when $dx/d\tau > 0$ and $h < 0$ when $dx/d\tau < 0$), we note that in order to obtain the trajectories in the *whole* state plane, we must take the upper half of the state plane in Figure 37 when $h > 0$, and we must take the lower half of the state plane in Figure 38 when $h < 0$. By "piecing together" these two half planes along the line $y = 0$, we obtain the state plane for the given problem (Figure 39).

By looking at the resulting picture of the state plane we see that the motion resulting from any initial condition will be a damped wave, since the motion of the representative point along a trajectory always tends

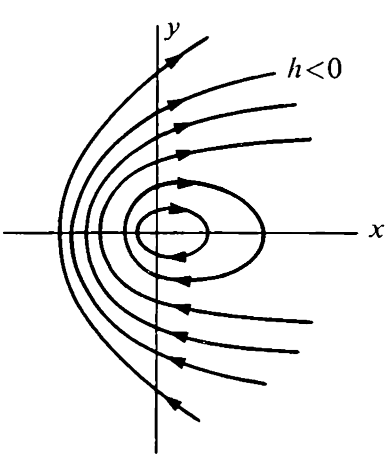


FIGURE 38

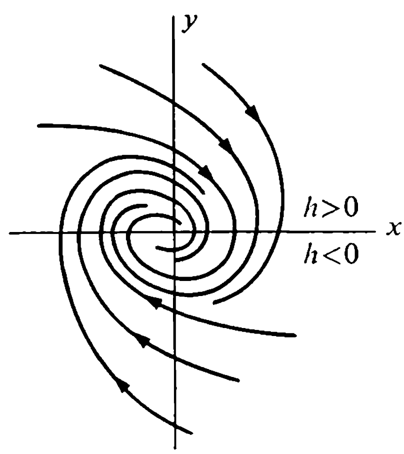


FIGURE 39

towards the equilibrium state at $x = 0$ and $y = 0$. Thus, both the position and velocity coordinates converge to zero.

By way of a more complicated example we shall examine the motion of a free gyroscope in Cardan suspension in the presence of Coulomb damping, [19, 20, 39, 40].

The equations of motion of a free gyroscope on a fixed (motionless) foundation, assuming that the rotor of the gyroscope rotates about its axis with a constant velocity, has the form [41]

$$\left.\begin{cases} J_B\ddot{\theta} + H\dot{\psi} = M_B, \\ J_C\ddot{\psi} - H\dot{\vartheta} = M_C, \end{cases}\right\} \tag{65}$$

where

J_B is the net moment of inertia of the gyroscope relative to the axis of rotation of the internal frame;
J_C is the net moment of inertia of the gyroscope relative to the axis of rotation of the external frame;
H is the kinetic moment of the gyroscope;
ψ is the angle of rotation of the external frame;
θ is the angle of rotation of the internal frame;
M_B is the moment of all forces relative to the axis of the internal frame;
M_C is the moment of all forces relative to the axis of the external frame.

Equations (65) are derived under the assumption that the angles ψ and θ are small, that the external and internal frames are mutually perpendicular, and that the axis of rotation of the external frame is vertical.

We shall consider the case where the center of gravity of the gyroscope is displaced along the axis of rotation of the gyroscope from the point of intersection of the Cardan axes. We shall designate the moment due to this unbalance by the letter M, and we shall assume that it is of the same order of magnitude as the moments due to Coulomb friction along the axes of the frames.

If both viscous and Coulomb damping are present, we have

$$M_B = M - \alpha\dot{\vartheta} + M_{BT},$$
$$M_C = -\beta\dot{\psi} + M_{CT},$$

where

$$M_{BT} = -L_B \operatorname{sign} \dot{\vartheta},^*$$
$$M_{CT} = -L_C \operatorname{sign} \dot{\psi},$$

* The symbol sign $\dot{\theta}$ means $(+1)$ or (-1) depending on whether $\dot{\vartheta} > 0$ or $\dot{\vartheta} < 0$. Usually sign $0 = 0$.

and where α and β are the coefficients of viscosity, L_B is the maximal moment due to Coulomb friction with respect to the axis of the internal frame, and L_C is the maximal moment due to Coulomb friction with respect to the axis of the external frame.

Moreover, we must keep in mind the fact that for $\dot{\theta} = 0$,

$$-L_B \leqslant M_{BT} \leqslant L_B,$$

and for $\dot{\psi} = 0$,

$$-L_C \leqslant M_{CT} \leqslant L_C.$$

We introduce new variables and notation:

$$x = \vartheta\sqrt{J_B}, \quad y = \psi\sqrt{J_C}, \quad \lambda = \frac{H}{\sqrt{J_B J_C}}, \quad n_1 = \frac{\alpha}{J_B},$$

$$n_2 = \frac{\beta}{J_C}, \quad m_1 = \frac{L_B}{\sqrt{J_B}}, \quad m_2 = \frac{L_C}{\sqrt{J_C}}, \quad M' = \frac{M}{\sqrt{J_B}}.$$

Then we can rewrite Equation (65) in the form

$$\begin{aligned} \ddot{x} + \lambda\dot{y} + n_1\dot{x} &= -m_1 \operatorname{sign} \dot{x} + M', \\ \ddot{y} - \lambda x + n_2\dot{y} &= -m_2 \operatorname{sign} \dot{y}. \end{aligned} \tag{66}$$

Equations (66) are a system of two differential equations of the second order. By means of a simple substitution we can reduce it to a system of equations of the first order. In fact, setting

$$u = \dot{x}, v = \dot{y},$$

we get

$$\begin{aligned} \dot{u} &= -\lambda v - n_1 u + M' - m_1 \operatorname{sign} u, \\ \dot{v} &= \lambda u - n_2 v - m_2 \operatorname{sign} v. \end{aligned} \tag{67}$$

Now the state plane (u,v) is a "plane of velocities." However, we can study the motion of a representative point in this particular state plane just as in the usual case.

The coordinate axes divide the uv-plane into four quadrants, in each of which the Equations (67) have a different form.

The differential equation of the integral curves on the uv-plane is

$$\frac{dv}{du} = \frac{\lambda u - n_2 v - m_2 \operatorname{sign} v}{-\lambda v - n_1 u + M' - m_1 \operatorname{sign} u}. \tag{68}$$

The singular points of this equation are determined by the expressions

$$\lambda u - n_2 v - m_2 \operatorname{sign} v = 0,$$

$$-\lambda v - n_1 u + M' - m_1 \operatorname{sign} u = 0$$

and have the coordinates

$$u_k = \frac{\lambda m_2 \operatorname{sign} v + n_2 (M' - m_1 \operatorname{sign} u)}{\lambda^2 + n_1 n_2},$$

$$v_k = \frac{\lambda (M' - m_1 \operatorname{sign} u) - n_1 m_2 \operatorname{sign} v}{\lambda^2 + n_1 n_2},$$

where $k = 1, 2, 3, 4$ is the number which corresponds to a quadrant, where in the first quadrant ($k = 1$) $u > 0$, $v > 0$, in the second ($k = 2$) $u < 0$, $v > 0$, in the third ($k = 3$) $u < 0$, $v < 0$, and in the fourth ($k = 4$) $u > 0$, $v < 0$.

We shall assume that $M' > m_1$, that is, that $M > L_B$.

Suppose that $\lambda = H/\sqrt{J_c J_b}$ is much greater than n_1, n_2, m_1, m_2, and M', we get

$$u_1 = \frac{\lambda m_2 + n_2 (M' - m_1)}{\lambda^2 + n_1 n_2} > 0,$$

$$v_1 = \frac{\lambda (M' - m_1) - n_1 m_2}{\lambda^2 + n_1 n_2} > 0,$$

Similarly,

$$u_2 > 0, v_2 > 0; u_3 < 0, v_3 > 0; u_4 < 0, v_4 > 0.$$

It follows that Equation (68) has only one singular point inside the first quadrant, which corresponds to the motion of the frame with angular velocities of

$$\left.\begin{aligned} \dot{\vartheta} &= \frac{\lambda m_2 + n_2 (M' - m_1)}{\sqrt{J_B} (\lambda^2 + n_1 n_2)} = \frac{H L_C + \beta (M - L_B)}{H^2 + \alpha\beta}, \\ \dot{\psi} &= \frac{\lambda (M' - m_1) - n_1 m_2}{\sqrt{J_C} (\lambda^2 + n_1 n_2)} = \frac{H(M - L_B) - \alpha L_C}{H^2 + \alpha\beta}. \end{aligned}\right\} \tag{69}$$

The equation which determines the character of the singular point has the form

$$s^2 + (n_1 + n_2) s + \lambda^2 + n_1 n_2 = 0.$$

For large λ ($\lambda > | n_1 - n_2 | /2$) this equation has complex roots with negative real parts. Consequently the singular point will be a stable focus.

We shall now determine the integral curves.

By rewriting Equation (68) with the change of variables

$$\xi = u - u_k, \eta = v - v_k,$$

we get

$$\frac{d\eta}{d\xi} = \frac{\lambda\xi - n_2\eta}{-\lambda\eta - n_1\xi}.$$

Integrating this equation we have

$$\xi^2 + 2h\xi\eta + \eta^2 = ce^{-\frac{2a}{\sqrt{1-h^2}} \arctan \frac{\eta + h\xi}{\xi\sqrt{1-h^2}}}.$$

Changing back to the variables u and v, we obtain the equations of the integral curves for each quadrant

$$\begin{aligned}(u - u_k)^2 + 2h(u - u_k)(v - v_k) + (v - v_k)^2 = \\ = ce^{-\frac{2a}{\sqrt{1-h^2}} \arctan \frac{(v - v_k) + h(u - u_k)}{(u - u_k)\sqrt{1-h^2}}},\end{aligned} \tag{70}$$

where $a = n_1 + n_2/2\lambda$; $h = n_1 - n_2/2\lambda$; and c is a constant of integration.

In each quadrant ($k = 1, \ldots, 4$) the curves of the family (70) are spirals whose focus lies at the point $u = u_k$, $v = v_k$.

By constructing the integral curves over the *whole* uv-plane, we can analyze the behavior of angular velocities as a function of the initial conditions, by following the motion of the representative point. The angular velocities of the resulting motion of the frames of the gyroscope will be a motion determined by formula (69).

There are segments on the u and v axes which are a part of integral curves. On the u-axis this part is from $-m_2/\lambda$ to m_2/λ, and on the v-axis it is from $M' - m_1/\lambda$ to $M' + m_1/\lambda$.

When the representative point is on this segment of the u-axis, the equations of motion (67) have the form

$$\dot{u} = -n_1 u + M' \mp m_1,$$

$$\dot{v} = \lambda u + \frac{M_{\text{CT}}}{\sqrt{J_{\text{C}}}} = 0,$$

where

$$-L_{\text{C}} \leqslant M_{\text{CT}} \leqslant L_{\text{C}}.$$

From these equations it follows that if a representative point has reached the segment which we are considering it will move along the u-axis in the positive direction up to the point $u = m_2/\lambda, v = 0$, where it will leave the u-axis, since we are supposing that $\lambda(M' - m_1) > n_1 m_2$, (that is, $v_1 > 0$).

When the representative point moves on the analogous segment of the v-axis, Equations (67) have the form:

$$\dot{u} = -\lambda v + M' + \frac{M_{BT}}{\sqrt{J_B}} = 0,$$

$$\dot{v} = -n_2 v - m_2 < 0,$$

where

$$-L_B \leqslant M_{BT} \leqslant L_B.$$

Motion along this segment will proceed in the direction of decreasing v up to the point $u = 0$, $v = M' - m_1/\lambda$, after which the representative point will leave the v-axis.

We shall now examine the motion of the axis of the gyroscope with zero initial velocities, that is, under the supposition that at the initial moment ($t = 0$)

$$\dot{\psi} = 0, \dot{\vartheta} = 0, \psi = 0, \vartheta = 0.$$

In this case the representative point at first moves along the u-axis up to the point $u = m_2/\lambda$, $v = 0$. This means that the external frame is motionless and the internal frame moves according to the equation

$$\dot{\vartheta} = \frac{M - L_B}{\alpha}\left(1 - e^{-\frac{\alpha}{J_B}t}\right).$$

The amount of time during which the external frame does not move is found from the condition that $u = m_2/\lambda$, that is, that

$$\frac{M - L_B}{\alpha}\left(1 - e^{-\frac{\alpha}{J_B}t_1}\right) = \frac{L_C}{H},$$

from whence

$$t_1 = J_B \frac{\ln\left[\dfrac{H(M - L_B)}{H(M - L_B) - \alpha L_C}\right]}{\alpha}.$$

The further motion of the representative point is along the spiral (70).

The construction of the family of curves (70) is fairly complicated. Therefore we shall study the special case where there is only Coulomb friction.

Then $n_1 = 0$ and $n_2 = 0$ and the family (70) reduces to a family of curves

$$(u - u_k)^2 + (v - v_k)^2 = c,$$

where in the first quadrant

$$u_1 = \frac{m_2}{\lambda}, \qquad v_1 = \frac{M' - m_1}{\lambda}.$$

In the other quadrants, we have correspondingly

$$u_2 = \frac{m_2}{\lambda}, \qquad v_2 = \frac{M' + m_1}{\lambda}; \qquad u_3 = -\frac{m_2}{\lambda},$$

$$v_3 = \frac{M' + m_1}{\lambda}; \qquad u_4 = -\frac{m_2}{\lambda}, \qquad v_4 = \frac{M' - m_1}{\lambda}.$$

If $M' > m_1$ the singular point will always lie in the first quadrant.

An integral curve located in the first quadrant and passing through the point $u = m_2/\lambda$, $v = 0$ has the following equation:

$$\left(u - \frac{m_2}{\lambda}\right)^2 + \left(v - \frac{M' - m_1}{\lambda}\right)^2 = \left(\frac{M' - m_1}{\lambda}\right)^2. \tag{71}$$

When

$$M < L_B + L_C \sqrt{\frac{J_B}{J_C}}$$

this circle does not intersect the v-axis.

When

$$M > L_B + L_C \sqrt{\frac{J_B}{J_C}}$$

there will be two points of intersection:

$$u = 0, \quad v = \sqrt{J_C}\,\frac{M - L_B + \sqrt{(M - L_B)^2 - \dfrac{J_B}{J_C} L_C^2}}{H};$$

$$u = 0, \quad v = \sqrt{J_C}\,\frac{M - L_B - \sqrt{(M - L_B)^2 - \dfrac{J_B}{J_C} L_C^2}}{H}.$$

The integral curve tangent to the v-axis at the point $u = 0$, $v = M' - m_1/\lambda$ has the equation

$$\left(u - \frac{m_2}{\lambda}\right)^2 + \left(v - \frac{M' - m_1}{\lambda}\right)^2 = \frac{m_2^2}{\lambda^2}.$$

When $M < L_B + L_C \sqrt{J_B/J_C}$ this circle intersects the u-axis at two points

$$u = \sqrt{J_B}\,\frac{L_C + \sqrt{L_C^2 - \dfrac{J_C}{J_B}(M - L_B)^2}}{H}, \quad v = 0;$$

$$u = \sqrt{J_B}\,\frac{L_C - \sqrt{L_C^2 - \dfrac{J_C}{J_B}(M - L_B)^2}}{H}, \quad v = 0.$$

When $M > L_B + L_C\sqrt{J_B/J_C}$ there will be no points of intersection with the u-axis.

The trajectories in the uv-plane are shown in Figure 40 when $M < L_B + L_C\sqrt{J_B/J_C}$ and in Figure 41 when $M > L_B + L_C\sqrt{J_B/J_C}$.

On that part of the u-axis which corresponds to motion along trajectories, the representative point moves in the direction of increasing u up to the point $u = m_2/\lambda$, $v = 0$. On the analogous part of the v-axis the representative point moves in the direction of decreasing v up to the point $u = 0$, $v = M' - M_1/\lambda$.

What is the physical picture of the motion of the gyroscope?

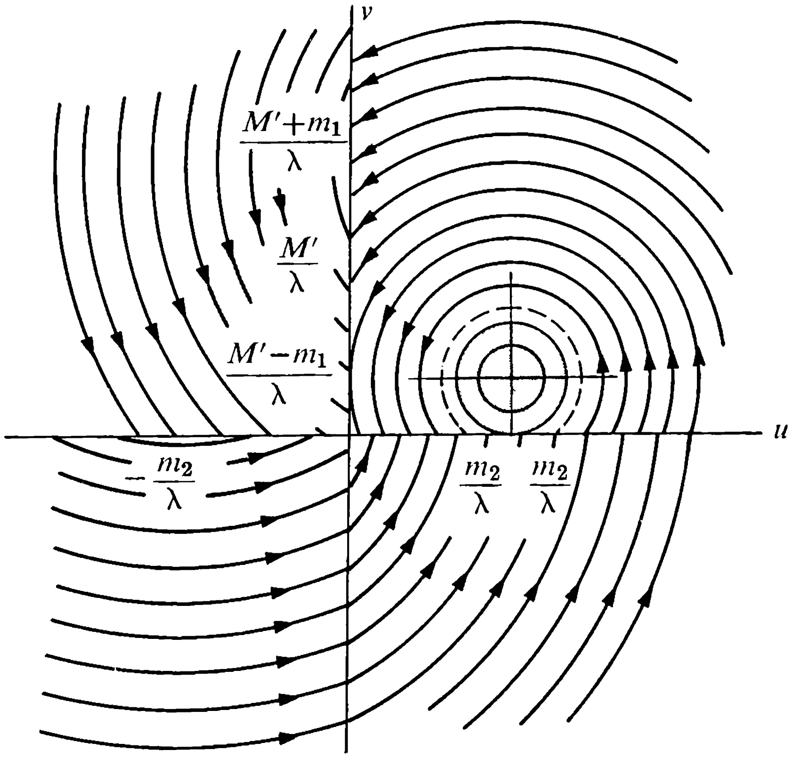


FIGURE 40

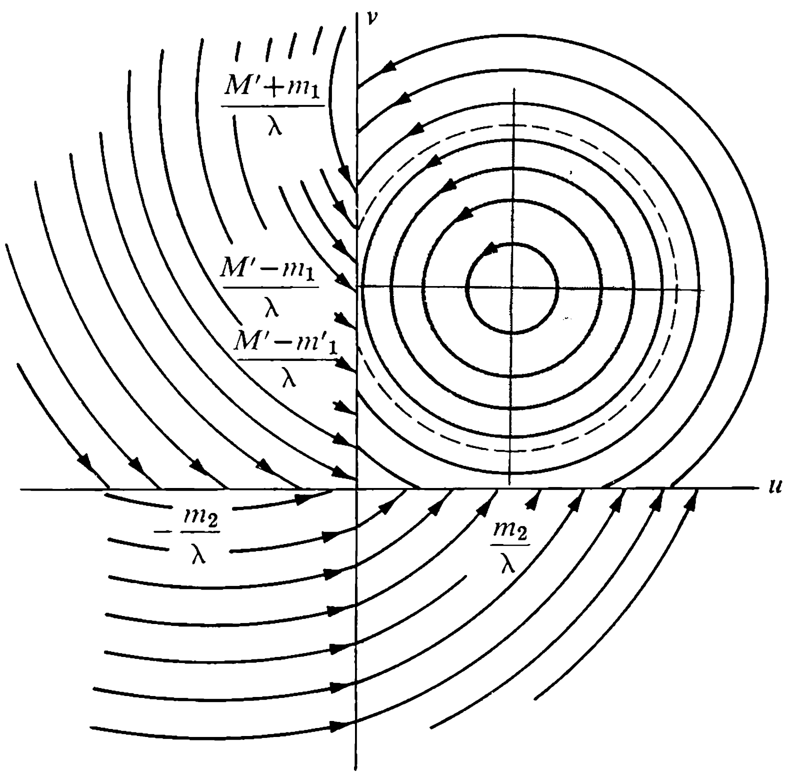


FIGURE 41

When the initial velocities correspond to the singular point

$$u_1 = \frac{m_2}{\lambda} = \sqrt{J_B}\,\frac{L_C}{H}, \quad v_1 = \frac{M' - m_1}{\lambda} = \sqrt{J_C}\,\frac{M - L_B}{H},$$

the motion of the external and internal frames will have the angular velocities

$$\dot{\psi} = \frac{M - L_B}{H}, \quad \dot{\vartheta} = \frac{L_C}{H}.$$

For initial velocities which correspond to points in the uv-plane located in the shaded regions of Figures 40 and 41, the equations of motion for the gyroscope have the following form:

$$\dot{u} = -\lambda v + M' - m_1,$$

$$\dot{v} = \lambda u - m_2.$$

The solution of these equations has the form

$$\dot{\vartheta} = \frac{L_C}{H} + c_1 \cos \lambda t + c_2 \sin \lambda t,$$

$$\dot{\psi} = \frac{M - L_B}{H} + \sqrt{\frac{J_B}{J_C}}(c_1 \sin \lambda t - c_2 \cos \lambda t),$$

where c_1 and c_2 are arbitrary constants of integration which are determined by the initial conditions of the corresponding shaded areas.

Therefore, under these initial conditions with uniform rotation of the internal and external frames with angular velocities equal to L_C/H and $M - L_B/H$, respectively, the nutational oscillations have the frequency λ.

Let us consider the motion resulting from zero initial conditions. In this case the representative point starts from the origin and moves along the u-axis. In the interval of time

$$t_1 = \frac{J_C L_C}{H(M - L_B)}$$

it reaches the point $u = m_2/\lambda$, $v = 0$.

Moreover, if

$$M < L_B + L_C \sqrt{\frac{J_B}{J_C}}$$

the representative point will then move along the circles (71), that is, a motion is established which is described by the Equations (Figure 40)

$$\left.\begin{aligned} \dot{\vartheta} &= \frac{L_C}{H} + \frac{M - L_B}{H} \sqrt{\frac{J_C}{J_B}} \sin \lambda t, \\ \dot{\psi} &= \frac{M - L_B}{\mathrm{H}}(1 - \cos \lambda t). \end{aligned}\right\} \qquad (72)$$

After an interval of time equal to $T = 2\pi/\lambda$, the angular velocity of the external frame will become zero. This takes place when the representative point goes through the point $u = m_2/\lambda$, $v = 0$. However, it should be noted that the cessation of motion of the external frame is not instantaneous, that is, the interval of time during the course of which the external frame is at rest will be nonzero. This is explained by the fact that the maximal moment L_C' due to Coulomb friction under stationary conditions is greater than the moment of L_C due to Coulomb friction when there is motion. That is, $L_C' > L_C$ and $m_2' > m_2$. Thus the representative point actually describes the path shown by the dotted line in Figure 40.

By making use of Figure 41, the reader can explain in an analogous manner the case

$$M > L_B + L_C \sqrt{\frac{J_B}{J_C}}.$$

In this case the motion of the gyroscope is described by the equations

$$\left.\begin{aligned} \vartheta &= \frac{L_C}{H}(1 - \cos\lambda t), \\ \dot{\psi} &= \frac{M - L_B}{H} - \frac{L_C}{H}\sqrt{\frac{J_B}{J_C}}\sin\lambda t. \end{aligned}\right\} \tag{73}$$

From Figures 40 and 41 it follows that for all other initial conditions which we have not considered the motion is described by Equations (72) if

$$M < L_B + L_C\sqrt{\frac{J_B}{J_C}}$$

and by Equations (73) if

$$M > L_B + L_C\sqrt{\frac{J_B}{J_C}}.$$

When $M = 0$ the graph of the subdivisions of the uv-plane into trajectories has the form shown in Figure 42. By examining this figure one can see that for arbitrary initial values of the angular velocities of the frames of the gyroscope, the representative point will necessarily arrive at a straight line segment of motion along one of the axes. Motion along these segments is directed towards the origin. Assume, for example, that the representative point reaches the segment of motions on the u-axis having started in the first quadrant. The equations of motion will then be

$$\dot{u} = -m_1 < 0,$$

$$\dot{v} = \lambda u - \frac{|M_{CT}|}{J_C} = 0,$$

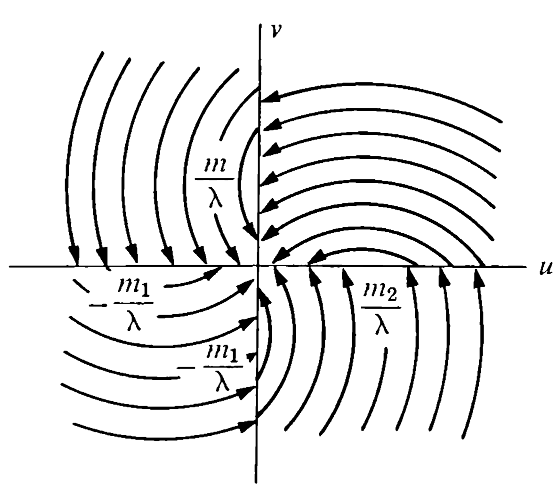


FIGURE 42

and it follows that the representative point moves towards the origin.

Thus for $M = 0$ the motion of the gyroscope for arbitrary initial conditions will be $\dot{\theta} = 0$, and $\dot{\psi} = 0$, that is, the axis of the gyroscope is stationary. We note that when $M \neq 0$ but $M < L_B$, we get an analogous graph.

GRAPHICAL SOLUTION OF THE EQUATIONS OF MOTION

In many instances, we require not only a graph of the trajectories on the state plane, but the variation of the x-coordinate with respect to time is also of interest. We shall now develop an approximate graphical method of determining the graph of the function $x(t)$, which is due to S. A. Dobrov [22].

On the state plane (x,y) (where $y = \dot{x}$) we consider a given trajectory $y = f(x)$ (Figure 43). We choose a point P on this trajectory corresponding to the location of the representative point at the initial moment of time ($t = 0$). Through this point we pass a line parallel to the y-axis to the point of intersection with the x-axis on the xt-plane (Figure 43). We denote this point of intersection by the letter A.

Then, we extend a segment from the point A along the x-axis whose length is equal to a unit of time. The segment is directed to the right if

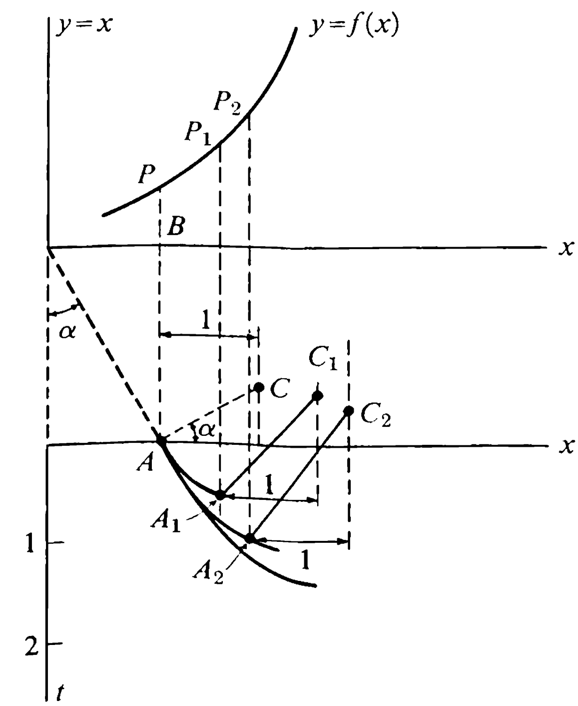


FIGURE 43

$y = \dot{x}$ increases with time, and to the left if $y = \dot{x}$ decreases with time. From the end point of this segment we draw a segment perpendicular to the x-axis, with length equal to $PB = \dot{x}$. We denote the end point of this segment by the letter C. This segment is drawn in the direction of the t-axis if $y = \dot{x}$ decreases with time, and in the opposite direction if $y = \dot{x}$ increases with time.

It is obvious that a tangent to the curve $x = x(t)$ at the point A will be perpendicular to the segment CA (Figure 43).

Through the point A we pass a circular arc centered at C and with radius CA. Then we choose a point P_1 near to the point P on the state plane. (P_1 is a point reached *after* P by the representative point moving along the trajectory.) The nearer P_1 is to the point P, the more accurate is our construction of the curve $x(t)$. Through the point P_1 we pass a line perpendicular to the x-axis up to its point of intersection A_1 with the circular arc which passes through the point A. We continue this construction just as we did at the point A, and we find a new point C_1. From this point C_1 we draw a circle, which begins in the middle of the arc AA_1.

We choose another point on the phase trajectory near to the point P_1 and continue the construction in an analogous manner.

5. The Method of Broken-Line Approximation. Self-Excited Oscillations

This method is one of the exact techniques for studying nonlinear systems. It is applicable when a nonlinear function in the equations of motion can be replaced by a function whose graph consists of a straight line segment. Then the motion of the system is described by linear differential equations in various regions of the state space. To each line segment in the broken-line approximation of the nonlinear function there will correspond one or more regions in the state plane where the system is governed by a linear differential equation.

The method is applied in the following way: given the initial conditions inside any of the regions, we solve the corresponding *linear* differential equation and obtain a solution which is valid over this region; then we determine the state of the system (its position and velocity) at the boundary of the neighboring region in which the motion is described by another linear differential equation; for the solution corresponding to the second region, we take as initial conditions the final state of the previous region; and so on.*

* The method can be applied to a more general case. For example, the state plane may be subdivided into regions in which the nonlinear functions entering the differential equation are such that the equation, although not linear, can be integrated. The principle of "piecing together" the solutions remains valid.

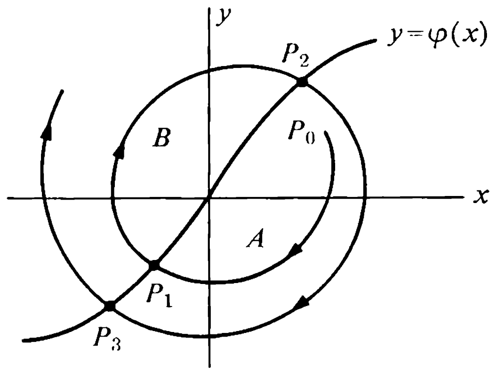


FIGURE 44

The method of broken-line approximation can be illustrated in a very picturesque way.

For example, let us assume that the curve $y = \varphi(x)$ (Figure 44) divides the plane into two regions, in each of which the trajectories are determined by two different linear differential equations.

The intersection $P_1\ (x_1,y_1)$ of the trajectory with the curve $y = \varphi\ (x)$ will be the initial point for the trajectory in the region B. The point $P_2(x_2,y_2)$ will be the initial point for the trajectory in the region A.

By piecing together the initial conditions we can follow the motion of the representative point over the whole plane and thereby determine the behavior of the dynamic system. Moreover, we can visualize the process in such a way that many trajectories are simultaneously determined.

Let us look at a concrete application of this method.

Let the dynamic system be described by the following differential equation

$$\ddot{\psi} + 2h\dot{x} + k^2x = F(\dot{x}), \tag{74}$$

where $F(\dot{x}) = F$ for $\dot{x} > 0$ and $F(\dot{x}) = -F$ for $\dot{x} < 0$.

By way of a physical system one can think of a pendulum, for example, with a viscous resistance, where there is a constant force which is always directed in the direction of motion.

In order to clarify the qualitative aspects of the motion we shall first apply the method of "piecing together" regions of the state plane.

Letting $\dot{x} = y$, we can rewrite Equation (74) in the form

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -2hy - k^2x + F(y). \tag{75}$$

Let us examine the half plane where $\dot{x} = y < 0$. Equations (75) in this case have the form

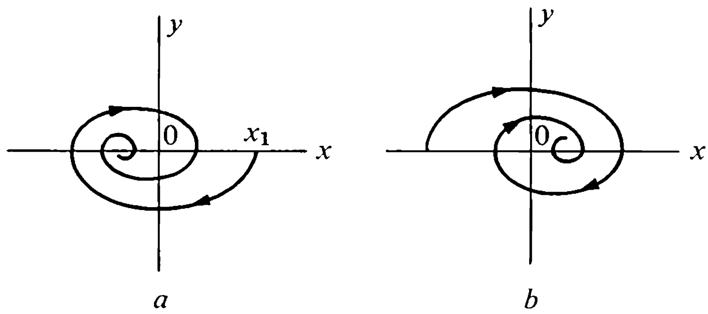


FIGURE 45

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -2hy - k^2x - F.$$

The differential equation of the trajectories has the form

$$\frac{dy}{dx} = \frac{-2hy - k^2x - F}{y}. \tag{76}$$

Here, the singular point is unique, and for $h < k$ it is a stable focus. The coordinates of the singular point are

$$x = -\frac{F}{k^2}, \quad y = 0.$$

If we suppose that Equation (76) is valid for all dx/dt, then the graph of the state plane will be as it is shown in Figure 45*a*.

For $\dot{x} = y > 0$, equation (76) has the form

$$\frac{dy}{dx} = \frac{-2hy - k^2x + F}{y}. \tag{77}$$

The singular point $x = F/k^2, y = 0$ will be a stable focus (h/k).

If Equation (77) were true for the whole plane, then the graph of the state plane would be as it is shown in Figure 45*b*.

By cutting both state planes along the line $y = 0$, we combine the upper half-plane of (Figure 45*b*) with the lower half of (Figure 45*a*).

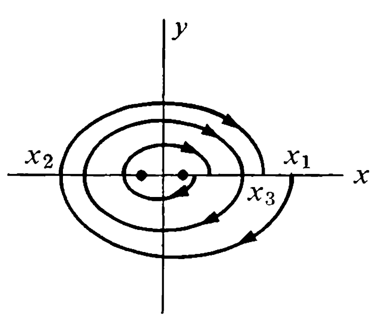


FIGURE 46

The character of the trajectories for this problem is shown in Figure 46. From the drawing one can see that for sufficiently large initial displacements, the motion of the system is a damped oscillation and, conversely, for sufficiently small initial displacements, the motion is an increasing oscillation.

As a result of this, it is natural to conclude that there exists a closed trajectory, around which all the other phase trajectories wind. This closed trajectory corresponds to periodic motion of the system, where the parameters of the periodic motion do not depend on the initial conditions and are determined by the parameters of the dynamic system.

For the region in which $\dot{x} < 0$, the solution of Equation (74) is

$$x = e^{-ht}(c_1 \cos k_1 t + c_2 \sin k_1 t) - \frac{F}{k^2},$$

where $k_1 = \sqrt{k^2 - h^2}$.

Since

$$\dot{x} = e^{-ht}[(-hc_1 + k_1 c_2) \cos k_1 t + (-hc_2 - k_1 c_1) \sin k_1 t],$$

it follows that for the initial conditions $t = 0$

$$x(0) = x_1, \quad y(0) = \dot{x}(0) = 0$$

we will have

$$c_1 = x_1 + \frac{F}{k^2}, \quad c_2 = \frac{h}{k_1}\left(x_1 + \frac{F}{k^2}\right)$$

and, consequently,

$$x = e^{-ht}\left(x_1 + \frac{F}{k^2}\right)\left(\cos k_1 t + \frac{h}{k_1} \sin k_1 t\right) - \frac{F}{k^2},$$

$$\dot{x} = -e^{-ht}\left(x_1 + \frac{F}{k^2}\right)\frac{k^2}{k_1^2} \sin k_1 t.$$

For the moment of time $t = \pi/k_1$,

$$x = x_2, \quad \dot{x} = 0,$$

where

$$x_2 = -\left[e^{-\frac{h\pi}{k_1}}\left(x_1 + \frac{F}{k^2}\right) + \frac{F}{k^2}\right]. \tag{78}$$

For the region where $\dot{x} > 0$, the solution of Equation (74) has the form

$$x = e^{-ht}(c_3 \cos k_1 t + c_4 \sin k_1 t) + \frac{F}{k^2}.$$

For the initial conditions $t = 0$, $x(0) = x_2$, $y(0) = 0$, we get

$$x = e^{-ht}\left(x_2 - \frac{F}{k^2}\right)\left(\cos k_1 t + \frac{h}{k_1}\sin k_1 t\right) + \frac{F}{k^2},$$

$$\dot{x} = -e^{-ht}\left(x_2 - \frac{F}{k^2}\right)\frac{k^2}{k_1}\sin k_1 t.$$

For $t = \pi/k_1$, $\dot{x}$ becomes zero, and x is equal to

$$x_3 = -e^{-\frac{h\pi}{k_1}}\left(x_2 - \frac{F}{k^2}\right) + \frac{F}{k^2}.$$

By denoting $T = \pi/k_1$ (the half period) and making use of Equation (78), we get

$$x_3 = x_1 e^{-2hT} + \frac{F}{k^2}(e^{-2hT} + 2e^{-hT} + 1)$$

$$= x_1 e^{-2hT} + \frac{F}{k^2}(1 + e^{-hT})^2.$$

For the resulting periodic motion, the condition

$$x_3 = x_1 = x_0,$$

must be fulfilled, that is,

$$x_0 = x_0 e^{-2hT} + \frac{F}{k^2}(1 + e^{-hT})^2,$$

which implies

$$x_0 = \frac{1 + e^{-hT}}{1 - e^{-hT}} \cdot \frac{F}{k^2}.$$

The resulting number x_0 is the maximal displacement of the pendulum from its position of equilibrium. This maximal displacement x_0 is called the *amplitude* of the periodic motion. We note that x_0 does not depend on the initial conditions but depends on the parameters of the system.

However, one fact in the determination of the amplitude x_0 is still missing. It is desirable to know for what initial conditions the periodic solution is valid and whether it will be stable. We can answer these questions by a special construction, the diagram of Koenigs and Lemeré. The connection between two successive maximal displacements with the same sign, that is, between x_1 and x_3, is graphically illustrated on this diagram. In Equation (79) x_3 can be thought of as a function of x_1. On the x_1x_3-plane the dependence between x_1 and x_3 is described by a straight line (Figure 47). The tangent of the angle of this line with the x_1-axis is

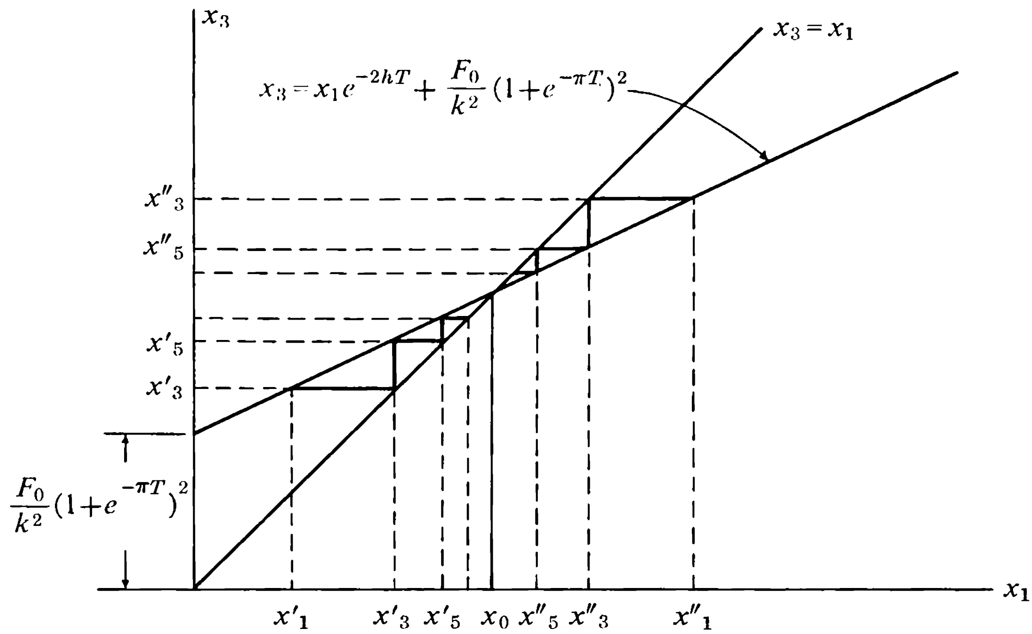


FIGURE 47

equal to $e^{-2hT} < 1$. Then we draw a second line, $x_1 = x_3$. The intersection of these two lines will determine the amplitude, $x_1 = x_3 = x_0$.

Let the initial displacement x'_1 be less than x_0. Then x'_3 will correspond to this displacement, and the succeeding maximal displacement x'_5 will correspond to x'_3. By continuing this construction we can see that no matter how small the initial displacements are, the amplitude of oscillation will increase until it reaches the stationary amplitude x_0. If the initial displacement x''_1 are larger than x_0, then we can find the displacement x''_3, and, consequently, the displacement x''_5 from Figure 47. From this construction it is clear that the oscillation will be damped until its amplitude achieves the stationary value x_0.

Thus, in spite of the presence of viscous damping, there will be a periodic motion in the present case. As is shown above, the amplitude of these sustained oscillations does not depend on the initial conditions, but is determined by the properties of the system. Such oscillations are said to be *self-excited.**

An isolated closed trajectory which corresponds to self-excited oscillations of the system is called a *limit cycle.* A limit cycle can be stable, unstable, or semistable.

If the trajectories which are close to the limiting cycle, both inside and outside, "wind" towards it, that is, if the representative point approaches

* The term "self-excited oscillations" was introduced by Academician A. A. Andronov.

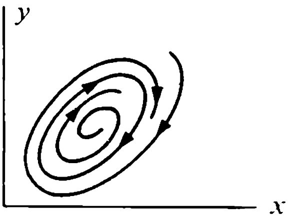


FIGURE 48

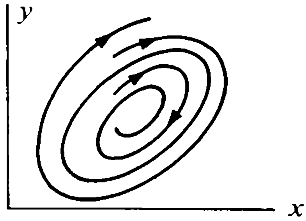


FIGURE 49

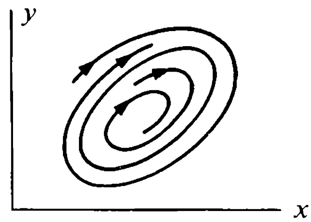


FIGURE 50

the limit cycle as $t \to \infty$, then this cycle is called stable and corresponds to self-excited oscillations.

If the trajectories wind away from the limit cycle both on the outside and the inside, that is if the representative point moves away from the limit cycle as $t \to \infty$, then it is called unstable and cannot be physically realized.

Finally, the limit cycle is semistable when the trajectories wind away from it on one side and towards it on the other.

In Figures 48, 49, and 50 there are graphs showing stable, unstable, and semistable limit cycles, respectively.

In this way, from a mathematical point of view one can define a system with one degree of freedom which is "capable of self-excited oscillations" as one whose equations of motion have one or several stable limit cycles.

From a physical point of view, systems capable of self-excited oscillations can be characterized, roughly speaking, as systems which contain sources of energy and, in which the input of energy expended to counteract the resistive forces in the system is regulated by the system itself.

A characteristic property of systems capable of self-excited oscillations is that in the steady state the amplitude of oscillation, generally speaking, does not depend on the initial conditions. This is the fundamental difference between limit cycles and periodic motions in conservative systems.

Although in the present example the period of self-excited oscillations turned out to be equal to the period of damped oscillations in a linear system, in general this period is determined by the properties of the nonlinear system itself. This is the basic difference between self-excited and forced oscillations.

6. The Method of Exact Transformations

The method of exact transformations is an exact method of solving nonlinear problems and is a generalization of the method of broken-line approximation. This method was worked out and applied to the solution of a whole series of important problems in the theory of clocks and the

theory of automatic control by Academician A. A. Andronov and his associates, N. N. Bautin, A. G. Mayer, G. S. Gorelik, Yu. I. Neymark, N. A. Zheletsov, and others.

In the method of broken-line approximation (Section 5) we constructed each trajectory separately. When using the method of exact transformations, one gets a subdivision of the whole state plane or the whole state space into trajectories.

We shall study the principles of this method with the aid of a simple example.*

Let the motion of a dynamic system be described by the following differential equation:

$$\ddot{x} + x = f(\dot{x}). \tag{80}$$

Here $f(x) = -2h_1x$ if $|x| > a$, and $f(\ddot{x}) = 2h_2\dot{x}$ if $|x| < a$, where $1 > h_1 > 0$ and $1 > h_2 > 0$.

Such a differential equation can be used to describe, for example, the motion of a harmonic oscillator in a medium with viscous damping under the condition that for $|x| < a$ there is a force acting on the oscillator which is proportional to the velocity and is acting in the direction of the motion.

If for all x, $f(\dot{x}) = -2h_1\dot{x}$, then the coordinate origin of the xy-phase plane would be a singular point which was a stable focus and all the phase trajectories would be spirals winding around this focus. If for all x, $f(\dot{x}) = 2h_2\dot{x}$, then the coordinate origin would be an unstable focus and all the phase trajectories would wind away from it.

Therefore, the state plane of Equation (80) is filled with pieces of spirals that are "glued together" at their ends along the lines $x = a$ and $x = -a$.

We shall call a sequence of pieces of spirals over which the representative point runs during the whole time of motion a trajectory. Let us look at the behavior of one such trajectory.

Let the initial conditions correspond to the point P_1 with coordinates $x = a$ and $y = u > 0$ (Figure 51). Up to the point P_2 the representative point will move along the spiral which corresponds to the equation

$$\ddot{x} + 2h_1\dot{x} + x = 0. \tag{81}$$

Then, up to the point P_3 the motion of the representative point will be along the spiral which corresponds to the equation

$$x - 2h_2\dot{x} + x = 0. \tag{82}$$

From the point P_3 to the point P_4 the representative point will again move according to Equation (81). Finally, from the point P_4 to the point

* This example was presented by N. N. Bautin.

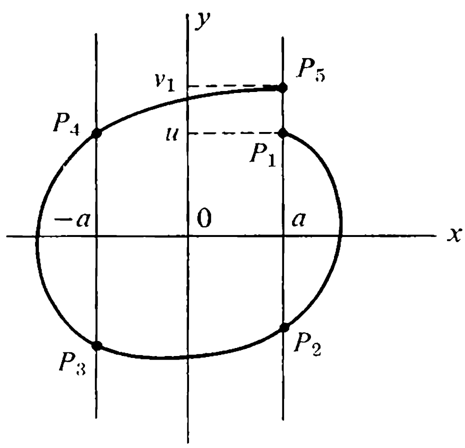


FIGURE 51

P_5 with coordinates $x = a, y = v_1$ the representative point will move once again according to Equation (82).

The objective of the method of exact transformations is to establish a dependence between v_1 and u. This dependence is expressed by a function which is called the sequence function. The sequence function determines the dependence between the succeeding and preceding positions of the representative point on the half line $x = a, y > 0$. We have already become acquainted with a sequence function in the preceding section. The expression (79) gave an explicit dependence between the preceding coordinate x_1 and the succeeding coordinate x_3 on the positive half of the x-axis.

In the present example, the sequence function will be in parametric form. It is worth mentioning that most of the time it is convenient to construct the sequence function in parametric form. A knowledge of the sequence function makes it possible to subdivide the whole state plane into its trajectories.

In this manner, the study of the behavior of a trajectory is equivalent to the study of the sequence of its points of intersection with the halfline, $x = a, y > 0$.

If $v_1 > u$, then motion along the trajectory in the resulting system corresponds to a divergent process. If $v_1 < u$, then the motion corresponds to a convergent process. Finally, if $v_1 = u$, then the trajectory is closed and corresponds to a periodic process.

The study of the structure of the subdivision of the whole phase plane is equivalent to the study of the structure of the transformation of the half line $x = a, y > 0$ into itself, which is carried out by the motion of the representation point along the pieces of the spirals.

Because of the symmetry of the phase plane relative to the origin, in the

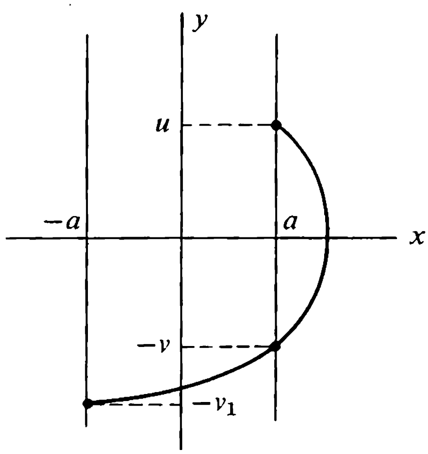


FIGURE 52

present example we can limit ourselves to a study of the transformation of the half line $x = a, y > 0$ into the half line $x = -a, y < 0$. This transformation, in turn, can be decomposed into two others: (1) the transformation of the half line $x = a, y > 0$ into the half line $x = a, y \leqslant 0$; and (2) the transformation of the half line $x = a, y \leqslant 0$ into the half line $x = -a, y \leqslant 0$ (Figure 52).

The solution of Equation (81) is

$$x = e^{-h_1 t}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t),$$

where $\omega_1^2 = 1 - h_1^2$.

Therefore

$$\dot{x} = e^{-h_1 t}\left[(-h_1 c_1 + c_2 \omega_1) \cos \omega_1 t - (h_1 c_2 + \omega_1 c_1) \sin \omega_1 t\right].$$

For the initial conditions $t = 0, x = a, y = \dot{x} = u > 0$, we get

$$c_1 = a, \quad c_2 = \frac{u + h_1 a}{\omega_1},$$

and, hence,

$$\left.\begin{aligned} x &= e^{-h_1 t}\left(a \cos \omega_1 t + \frac{a h_1 + u}{\omega_1} \sin \omega_1 t\right), \\ \dot{x} = y &= e^{-h_1 t}\left(u \cos \omega_1 t - \frac{h_1 u + a}{\omega_1} \sin \omega_1 t\right). \end{aligned}\right\} \tag{83}$$

The representative point which at the initial moment is on the line $x = a, y > 0$ will, after an interval of time $t = t_0$ arrive at the half line $x = a, y \leqslant 0$. We shall call this mapping the transformation S^+.

For $t = t_0 = \tau/\omega_1$, let

$$x = a, \quad y = -v\,(v > 0). \tag{84}$$

We shall find v as a function of u. To this purpose we put the conditions (84) into the Equation (83):

$$a = e^{-h_1\frac{\tau}{\omega_1}}\left(a\cos\tau + \frac{ah_1 + u}{\omega_1}\sin\tau\right),$$

$$-v = e^{-h_1\frac{\tau}{\omega_1}}\left(u\cos\tau - \frac{h_1 u + a}{\omega_1}\sin\tau\right).$$

By introducing the notation

$$k_1 = \frac{h_1}{\omega_1} = \frac{h_1}{\sqrt{1 - h_1^2}}$$

and solving these equations for u and v, we get

$$\left.\begin{aligned} u &= a\quad \frac{e^{k_1\tau} - \cos\tau - k_1\sin\tau}{\sqrt{1 + k_1^2}\sin\tau} > 0,\\ v &= a\quad \frac{e^{-k_1\tau} - \cos\tau + k_1\sin\tau}{\sqrt{1 + k_1^2}\sin\tau} > 0. \end{aligned}\right\} \tag{85}$$

We note that because of Equation (83),

$$0 \leqslant \tau \leqslant \pi.$$

The expressions in Equation (85) are parametric expressions for the transformation S^+ because they show the dependence of u on v by the parameter τ.

We shall denote the mapping of the point $x = a, y = -v$ into the point $x = -a, y = -v_1$ by E^+.

The solution of Equation (82) is

$$x = e^{h_2 t}(c_3\cos\omega_2 t + c_4\sin\omega_2 t)$$

and, hence,

$$\dot{x} = y = e^{h_2 t}[(h_2c_3 + c_4\omega_2)\cos\omega_2 t + (h_2c_4 - \omega_2c_3)\sin\omega_2 t],$$

where $\omega_2^2 = 1 - h_2^2$.

With the initial conditions $t = 0, x = a, y = -v$ we get

$$c_3 = a, \quad c_4 = -\frac{v + ha}{\omega_2}.$$

Consequently,

$$\left.\begin{aligned} x &= e^{h_2 t}\left(a \cos \omega_2 t - \frac{v + h_2 a}{\omega_2} \sin \omega_2 t\right), \\ \dot{x} = y &= -e^{h_2 t}\left(v \cos \omega_2 t + \frac{h_2 v + a}{\omega_2} \sin \omega_2 t\right). \end{aligned}\right\} \tag{86}$$

Now, let the representative point, located on the half line $x = a$, $y \leqslant 0$ at the time $t = 0$ move to a point on the half line $x = -a$, $y \leqslant 0$ with coordinates $x = -a$, $y = -v_1 < 0$ in an interval of time $t = t'_0 = \tau'/\omega_2$.

We shall find the parametric expressions for this transformation (the transformation E^+).

By using Equation (86) we get

$$-a = e^{k_2 \tau'}\left(a \cos \tau' - \frac{v + h_2 a}{\omega_2} \sin \tau'\right),$$

$$-v_1 = -e^{k_2 \tau'}\left(v \cos \tau' + \frac{h_2 v + a}{\omega_2} \sin \tau'\right),$$

where

$$k_2 = \frac{h_2}{\omega_2} = \frac{h_2}{\sqrt{1 - h_2^2}} > 0.$$

Thus, we have the desired parametric expression for E^+:

$$\left.\begin{aligned} v &= a\,\frac{e^{-k_2 \tau'} + \cos \tau' - k_2 \sin \tau'}{\sqrt{1 + k_2^2}\,\sin \tau'} > 0; \\ v_1 &= a\,\frac{e^{k_2 \tau'} + \cos \tau' + k_2 \sin \tau'}{\sqrt{1 + k_2^2}\,\sin \tau'} > 0. \end{aligned}\right\} \tag{87}$$

We note that

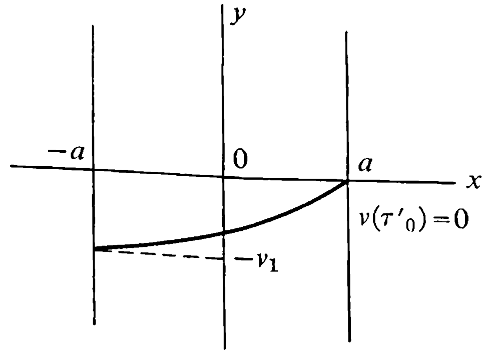


FIGURE 53

$$0 \leqslant \tau' \leqslant \tau_0' \leqslant \pi,$$

where τ_0' is determined by the equation $v(\tau_0') = 0$, since $k_2 = h_2/\omega_2 > 0$.

The trajectory for $\tau' = \tau_0'$ is shown in Figure 53.

From Equation (80) we can likewise determine the transformations S^- and E^-, which are symmetric to S^+ and E^+ in the sense that if points to be mapped by the transformations S^+ and E^+ are symmetric, then the transformed points will be symmetric.

In order to clarify the behavior of the trajectories, we study the behavior of the curve (85) on the uv-plane and the behavior of the curve (87) on the v_1v-plane.

Let us note that the parametric form of these curves is particularly convenient for studying their behavior.

For the curve determined by the Equations (85), we have

$$u(0) = v(0) = 0, \quad u(\pi) = v(\pi) = \infty.$$

Since

$$\frac{du}{d\tau} = \frac{a}{\sqrt{1+k_1^2}} \cdot \frac{1 + e^{k_1\tau}(k_1 \sin\tau - \cos\tau)}{\sin^2\tau} > 0$$

and

$$\frac{dv}{d\tau} = \frac{a}{\sqrt{1+k_1^2}} \cdot \frac{1 - e^{-k_1\tau}(k_1 \sin\tau + \cos\tau)}{\sin^2\tau} > 0,$$

and

$$\frac{dv}{d\tau} = \frac{a}{\sqrt{1+k_1^2}} \cdot \frac{1 - e^{-k_1\tau}(k_1 \sin\tau + \cos\tau)}{\sin^2\tau} > 0,$$

it follows that

$$\frac{du}{dv} = \frac{1 + e^{k_1\tau}(k_1 \sin\tau - \cos\tau)}{1 - e^{-k_1\tau}(k_1 \sin\tau + \cos\tau)} > 0.$$

Therefore,

$$\left(\frac{du}{dv}\right)_{\tau=0} = 1, \quad \left(\frac{du}{dv}\right)_{\tau=\pi} = \frac{1 + e^{k_1\pi}}{1 + e^{-k_1\pi}} = e^{k_1\pi}. \tag{88}$$

With the aid of the expression

$$\frac{d^2u}{dv^2} = \frac{d}{d\tau}\left(\frac{du}{dv}\right)\frac{d\tau}{dv},$$

we find that

$$\frac{d^2u}{dv^2} = \frac{(1 + k_1^2)^{\frac{3}{2}} \sin^3 \tau \, (e^{k_1\tau} - e^{-k_1\tau} - 2k_1 \sin \tau)}{a\,[1 - e^{-k_1\tau}(k_1 \sin \tau + \cos \tau)]^3}. \tag{89}$$

The sign of d^2u/dv^2 is the same as the sign of the function

$$f(\tau) = e^{k_1\tau} - e^{-k_1\tau} - 2k_1 \sin \tau.$$

Since

$$f'(\tau) = k_1 e^{k_1\tau} + k_1 e^{-k_1\tau} - 2k_1 \cos \tau,$$

it follows that for $\tau = 0, f'(0) = 0$.

For $0 \leqslant \tau \leqslant \pi$, the second derivative is

$$f''(\tau) = k_1^2 (e^{k_1\tau} - e^{-k_1\tau}) + 2k_1 \sin \tau > 0,$$

and, consequently, $f(\tau) > 0$ and $d^2u/dv^2 > 0$.

From Equations (88) and (89) it follows that

$$\frac{du}{dv} > 1.$$

We shall find the asymptote of the curve (85). We will find the equation of the asymptote in the form

$$u = kv + c,$$

and, hence,

$$k = \lim_{\tau \to \pi} \left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)_{\tau = \pi} = e^{k_1\pi},$$

$$c = \lim_{\tau \to \pi} [u - e^{k_1\pi}v] = -2k_1 (1 + e^{k_1\pi}) \frac{a}{\sqrt{1 + k_1^2}}.$$

Therefore, the equation of the asymptote has the form

$$u = ve^{k_1\pi} - 2k_1 (1 + e^{k_1\pi}) \frac{a}{\sqrt{1 + k_1^2}}.$$

The form of the curve (85) is shown in Figure 54.

We go on to the study of the curve (87). For Equation (87) we have

$$v_1(0) = v(0) = \infty,\ v(\tau_0') = 0,\ v_1(\tau_0') > 0.$$

In fact, if $v(\tau_0') = 0$, then

$$e^{-k_2\tau_0'} + \cos \tau_0' - k_2 \sin \tau_0' = 0$$

and, consequently,

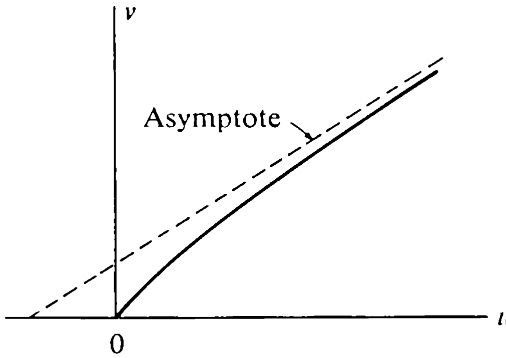


FIGURE 54

$$v_1 = a\frac{e^{k_2\tau_0'} + \cos\tau_0' + k_2\sin\tau_0'}{\sqrt{1+k_2^2}\sin\tau_0'}$$

$$= a\frac{e^{k_2\tau_0'} - e^{-k_2\tau_0'} + 2k_2\sin\tau_0'}{\sqrt{1+k_2^2}\sin\tau_0'} > 0.$$

Proceding as in the case of the curve (85), we find that

$$\frac{dv_1}{dv} = \frac{1 + e^{k_2\tau'}(\cos\tau' - k_2\sin\tau')}{1 + e^{-k_2\tau'}(\cos\tau' + k_2\sin\tau')}.$$

From this expression it follows that

$$\left(\frac{dv_1}{dv}\right)_{\tau'=0} = 1, \ \left(\frac{dv_1}{dv}\right)_{\tau'=\tau'_0} = 0, \tag{90}$$

since $v(\tau_0') = 0$.

Moreover, we have

$$\frac{d^2v_1}{dv^2} = \frac{(1+k_2^2)^{\frac{3}{2}}\sin^3\tau'\,(e^{k_2\tau'} - e^{-k_2\tau'} + 2k_2\sin\tau')}{a\,[1 + e^{-k_2\tau'}(\cos\tau' + k_2\sin\tau')]^3}. \tag{91}$$

The sign of the expression (91) is determined by the sign of the function

$$f_1(\tau') = e^{k_2\tau'} - e^{-k_2\tau'} + 2k_2\sin\tau',$$

but, since $k_2 > 0, f_1(\tau') > 0$, and it follows that $d^2v_1/dv^2 > 0$.

From Equations (90) and (91) we get

$$\frac{dv_1}{dv} < 1.$$

The equation for the asymptote of the curve (87) is

$$v_1 = v + 4k_2\frac{a}{\sqrt{1+k_2^2}}.$$

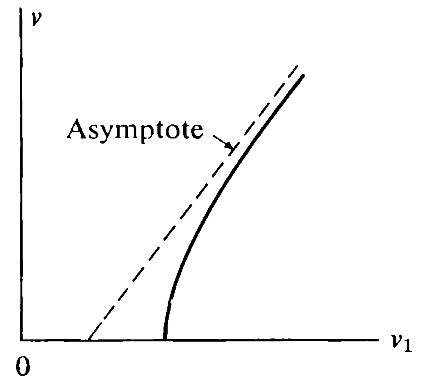


FIGURE 55

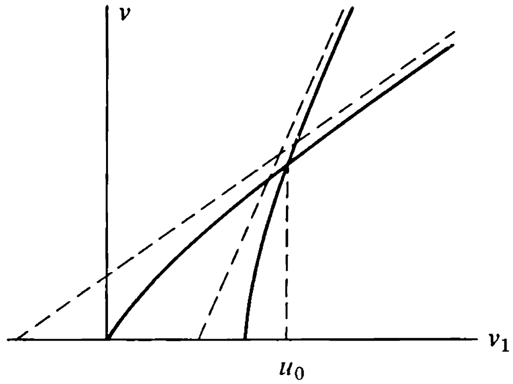


FIGURE 56

The curve (87) is shown in Figure 55.

Let us look at the curves (85) and (87) on the same plane by putting v on one of the axes and u and v_1 on the other. Since $k_1 > 0$ and $k_2 > 0$, the curves (85) and (87) have a unique point of intersection. One can verify the existence of this point by examining the behavior of the asymptote and the values of $u(0)$, $v(0)$, $v_1(\tau_0')$, and $v(\tau_0')$. There cannot be more than one point of intersection because, for two neighboring points of intersection, the quotient

$$\frac{du}{dv} \Big/ \frac{dv_1}{dv}$$

must pass from a value greater than one to a value less than one. But this is impossible because $du/dv > 1$ and $dv_1/dv < 1$.

The curves (85) and (87) are shown in Figure 56. The point of intersection of the curves (85) and (87) gives the value u_0 of the point on the half line $x = a$, $y > 0$, which is invariant, that is, the point which does not change relative to the transformation $T^* = E^- S^- E^+ S^+$, because the transformation $E^+ S^+$ maps the point $P_0(a,u_0)$ into a point symmetric to it relative to the origin. Owing to the characteristics of the transformations E^- and S^- already mentioned, the transformation T^* maps $P_0(a,u_0)$ into itself.

We shall investigate the stability of the point P_0. Let u' be an initial point on the half line $x = a$, $y > 0$ near to the point u_0 and let u'' be its image by the transformation T^*. The transformations S^+ and E^+ can be written in the form

$$(S^+) \qquad v - v_0 = \left(\frac{dv}{du'}\right)_{u'=u_0} (u' - u_0)^1 + \cdots,$$

because $v(u_0) = v_0$ where v_0 is the image of u_0 by the transformation S^+;

$$(E^+) \qquad v_1 - u_0 = \left(\frac{dv_1}{dv}\right)_{v_1=u_0} (v - v_0) + \cdots,$$

because $v_1(v_0) = u_0$.*

For the transformation $S^+ E^+$ we get

$$(S^+ E^+) \qquad v_1 - u_0 = \left(\frac{dv_1}{dv}\right)_{v_1=u_0} \left(\frac{dv}{du'}\right)_{u'=u_0} (u' - u_0) + \cdots. \tag{92}$$

Analogous expressions for the transformation $E^- S^-$ can be obtained from Equation (92) by replacing u' by v_1 on the right side and v_1 by u'' on the left side, that is,

$$u'' - u_0 = \left(\frac{dv_1}{dv}\right)_{v_1=u_0} \left(\frac{dv}{du'}\right)_{u'=u_0} (v_1 - u_0) + \cdots.$$

Consequently, for the transformation T^* we have

$$u'' - u_0 = \left(\frac{dv_1}{dv}\right)^2_{v_1=u_0} \left(\frac{dv}{du'}\right)^2_{u'=u_0} (u' - u_0) + \cdots.$$

Stability depends on the quantity

$$\lambda = \left(\frac{dv_1}{dv}\right)^2 \left(\frac{dv}{du}\right)^2,$$

computed at the invariant point.

The invariant point and its corresponding periodic solution are stable if $\lambda < 1$ and unstable if $\lambda > 1$.

In fact, since u' is the coordinate of the initial point which is near to u_0 on the half line $x = a, y > 0$ and u'' is its image under the transformation T^*, it follows that for $\lambda < 1$, $u'' - u_0 < u' - u_0$. This means that if the representative point is displaced from the closed trajectory which passes through the point u_0, which is invariant under the transformation T^*, then the representative point will converge towards the closed trajectory, that is, the periodic motion will be stable.

Since $du/dv > 1$ and $dv_1/dv < 1$, the quantity $\lambda < 1$ and, consequently, the invariant point and the periodic motion are stable (that is there is a stable limit cycle).

The stability of the limit cycle can be determined directly from a dia-

* We are expanding the function $v = v(u')$ in a power series in the variable $u' - u_0$ and are dropping terms with a power greater than the first degree in the variable $u' - u_0$.

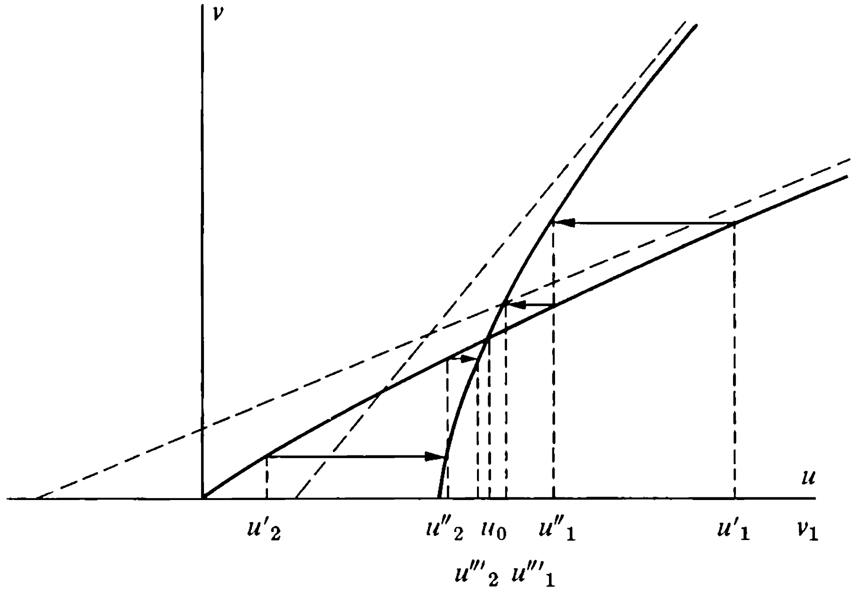


FIGURE 57

gram which is analogous to the diagram of Koenigs and Lemeré (Figure 57).

The period T of the periodic motion is given by the formula

$$T = 2\,(\tau_1 + \tau_1'),$$

where τ_1 and τ_1' are determined from the equations

$$u\,(\tau) = v_1\,(\tau') \text{ and } v\,(\tau) = v\,(\tau').$$

In their expanded form these equations are as follows:

$$\frac{e^{k_1\tau} - \cos\tau - k_1 \sin\tau}{\sqrt{1 + k_1^2}\sin\tau} = \frac{e^{k_2\tau'} + \cos\tau' + k_2 \sin\tau'}{\sqrt{1 + k_2^2}\sin\tau'},$$

$$\frac{e^{-k_1\tau} - \cos\tau + k_1 \sin\tau}{\sqrt{1 + k_1^2}\sin\tau} = \frac{e^{-k_2\tau'} + \cos\tau' - k_2 \sin\tau'}{\sqrt{1 + k_2^2}\sin\tau'}.$$

The general form of the state plane is shown in Figure 58.

We shall look at two special cases.

1) $k_1 = 0$; in this case $\tau_1 = \pi, \tau_1' = 0$, and the limit cycle may be thought of as being at infinity since $u_0 = \infty$.

2) $k_2 = 0$; then $\tau_1 = 0$, and $\tau_1' = \pi$. In this case $u_0 = 0$, that is, the limit cycle degenerates to the boundary of the region $R \leqq a$ (which is filled with circles of radius $R \leqslant a$), and spirals wind towards this region from the outside.

The state planes for the cases $k_1 = 0$ and $k_2 = 0$ are shown in Figures 59 and 60, respectively.

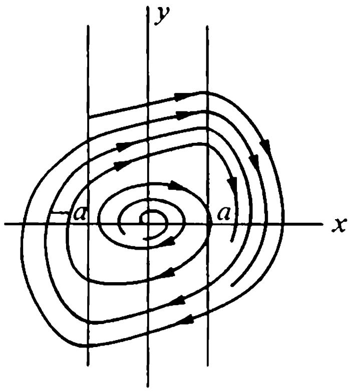


FIGURE 58

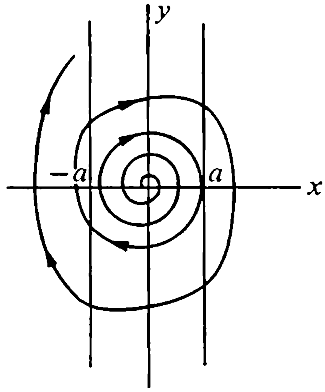


FIGURE 59

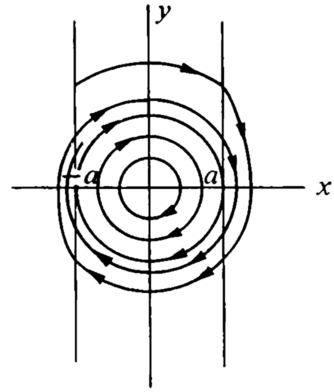


FIGURE 60

7. The Method of Slowly Varying Parameters (*van der Pol's Method*)

There are several approximation techniques for the quantitative study of nonlinear systems, for example, the methods of Poincaré and A. N. Krylov, the method of slowly varying parameter (of van der Pol), the methods of B. V. Bulgakov, N. M. Krylov, N. N. Bogolyubov, L. S. Goldfarb, and E. P. Popov, the energy method of Teodorchik, the method of A. I. Lur'e and A. I. Chekmarev, which is based on B. G. Galerkin's variational equation, the method of I. G. Malkin, and so on.

We will examine in detail the method of slowly varying parameters since one can use the contents of the preceding sections to a significant extent in applying this method.

Let the equation of motion for a dynamic system have the form

$$\frac{d^2x}{dt^2} + k^2x = \mu f_1\left(x, \frac{dx}{dt}\right), \tag{93}$$

where f_1 $(x, dx/dt)$ is a nonlinear function of x and dx/dt.

We shall suppose that this dynamic system is nearly linear and conservative. The coefficient μ (which is usually dimensionless) will be a small parameter which characterizes how close the resulting system is to a linear, conservative one. We choose $\mu > 0$.

We introduce the "dimensionless" time $\tau = kt$. Then, because

$$\frac{dx}{dt} = \frac{dx}{d\tau}k$$

and

$$\frac{d^2x}{dt^2} = \frac{d^2x}{d\tau^2}k^2,$$

Equation (93) takes the form

$$\frac{d^2x}{d\tau^2} + x = \mu f\left(x, \frac{dx}{d\tau}\right), \tag{94}$$

where

$$f\left(x, \frac{dx}{d\tau}\right) = \frac{1}{k^2} f_1\left(x, \frac{dx}{d\tau} k\right).$$

By setting $y = dx/d\tau$, we get

$$\frac{dy}{d\tau} = -x + \mu f(x,y),$$

and

$$\frac{dx}{d\tau} = y.$$

The equation of the integral curves is

$$\frac{dy}{dx} = \frac{-x + \mu f(x,y)}{y}. \tag{95}$$

For $\mu = 0$ this system is linear and conservative. The trajectories are concentric circles which surround the equilibrium state $x = 0, y = 0$.

The solution of Equation (94) in the case $\mu = 0$ is

$$x = a \cos \tau + b \sin \tau, \tag{96}$$

where a and b are arbitrary constants of integration.

Our problem is to clarify the question of how the graph of the collection of trajectories on the xy-plane changes if μ is sufficiently small but different from zero.

We will try to find a solution of Equation (94) in the form of Equation (96) for $\mu \neq 0$, but where a and b are functions of time. It is obvious that for a given x the functions a and b are not completely defined by this requirement; we impose therefore the further restriction that the derivative of x with respect to τ must have the same form as it does when a and b are constants. Thus,

$$\left.\begin{aligned} x &= a \cos \tau + b \sin \tau, \\ \frac{dx}{d\tau} &= -a \sin \tau + b \cos \tau + \frac{da}{d\tau} \cos \tau + \frac{db}{d\tau} \sin \tau. \end{aligned}\right\} \tag{97}$$

But because of the restriction imposed on a and b,

$$\frac{dx}{d\tau} = -a \sin \tau + b \cos \tau; \tag{98}$$

consequently,

$$\frac{da}{d\tau} \cos \tau + \frac{db}{d\tau} \sin \tau = 0. \tag{99}$$

The second derivative of x with respect to τ is

$$\frac{d^2x}{d\tau^2} = -a \cos \tau - b \sin \tau - \frac{da}{d\tau} \sin \tau + \frac{db}{d\tau} \cos \tau. \tag{100}$$

Substituting Equations (97), (98), and (100) into Equation (94), we get

$$-\frac{da}{d\tau} \sin \tau + \frac{db}{d\tau} \cos \tau = \mu f(a \cos \tau + b \sin \tau, -a \sin \tau + b \cos \tau). \tag{101}$$

If we apply the condition in Equation (99) to this equation, we get a system of equations which determines $da/d\tau$ and $db/d\tau$:

$$\frac{da}{d\tau} \cos \tau + \frac{db}{d\tau} \sin \tau = 0,$$

$$-\frac{da}{d\tau} \sin \tau + \frac{db}{d\tau} \cos \tau = \mu f(a \cos \tau + b \sin \tau, -a \sin \tau + b \cos \tau).$$

By solving this system for $da/d\tau$ and $db/d\tau$, we have

$$\left.\begin{aligned} \frac{da}{d\tau} &= -\mu f(a \cos \tau + b \sin \tau, -a \sin \tau + b \cos \tau) \sin \tau, \\ \frac{db}{d\tau} &= \mu f(a \cos \tau + b \sin \tau, -a \sin \tau + b \cos \tau) \cos \tau. \end{aligned}\right\} \tag{102}$$

We note that Equation (102) is Equation (94) transformed into new coordinates. As is clear from Equation (102) this system is not autonomous even though the initial equation was autonomous.

From the expressions of Equation (102) it follows that for sufficiently small μ the derivatives $da/d\tau$ and $db/d\tau$ are arbitrarily small and consequently $a(\tau)$ and $b(\tau)$ will be slowly varying functions with respect to time.

Due to this consideration we make the further assumption that the functions $a(\tau)$ and $b(\tau)$ change so slowly that their variation in the course of one period of the initial dynamic system ($\mu = 0$) is negligible. We shall also suppose that $da/d\tau$ and $db/d\tau$, which are of the order of μ, are constant during the interval of one period. Then, multiplying both sides of the equation by $d\tau$ and integrating from 0 to 2π, we get

$$\left.\begin{aligned}\frac{da}{d\tau} &= -\frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau,\ -a\sin\tau + b\cos\tau)\sin\tau\, d\tau,\\ \frac{db}{d\tau} &= \frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau,\ -a\sin\tau + b\cos\tau)\cos\tau\, d\tau.\end{aligned}\right\} \quad (103)^*$$

Essentially, these equations differ from Equation (102) in that their right sides are the mean values of the right sides of Equation (102) over the period of length 2π.

We rewrite the Equations (103) in the form

$$\left.\begin{aligned}\frac{da}{d\tau} &= \mu P(a,b),\\ \frac{db}{d\tau} &= \mu Q(a,b),\end{aligned}\right\} \quad (104)$$

where

$$P(a,b) = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau,\ -a\sin\tau + b\cos\tau)\sin\tau\, d\tau,$$

$$Q(a,b) = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau,\ -a\sin\tau + b\cos\tau)\cos\tau\, d\tau.$$

The equations (104) are called the "shortened" forms or van der Pol's equations.

We shall study the *ab*-plane. It is obvious from Equation (97) that the equilibrium state on the *ab*-plane at $a = 0$, $b = 0$ corresponds to the equilibrium state of the original system at $x = 0, y = 0$. The equilibrium states on the *ab*-plane for which $a \neq 0$, $b \neq 0$ correspond to periodic motions in the original system.

By studying the equilibrium states and the distribution of the trajectories on the *ab*-plane, one can obtain information about the possible motions of the original system. This technique was first applied by A. A. Andronov.

By changing to polar coordinates in Equation (104) we can separate variables. Let

---

*In the computation of these integrals a and b are considered constant.

$$\begin{aligned} a &= \rho \cos \vartheta, \\ b &= \rho \sin \vartheta, \end{aligned} \tag{105}$$

then

$$\frac{da}{d\tau} = \frac{d\rho}{d\tau}\cos\vartheta - \rho\sin\vartheta\frac{d\vartheta}{d\tau}, \frac{db}{d\tau} = \frac{d\rho}{d\tau}\sin\vartheta + \rho\cos\vartheta\frac{d\vartheta}{d\tau},$$

$$\begin{aligned} a\cos\tau + b\sin\tau &= \rho\cos(\tau - \vartheta), \\ -a\sin\tau + b\cos\tau &= -\rho\sin(\tau - \vartheta). \end{aligned}$$

By substituting these expressions in the Equations (104) we get

$$\frac{d\rho}{d\tau}\cos\vartheta - \rho\sin\vartheta\frac{d\vartheta}{d\tau} = -\frac{\mu}{2\pi}\int_0^{2\pi} f[\rho\cos(\tau-\vartheta), -\rho\sin(\tau-\vartheta)] \times \sin\tau\, d\tau;$$

and, hence

$$\frac{d\rho}{d\tau}\sin\vartheta + \rho\cos\vartheta\frac{d\vartheta}{d\tau} = \frac{\mu}{2\pi}\int_0^{2\pi} f[\rho\cos(\tau-\vartheta), -\rho\sin(\tau-\vartheta)] \times \cos\tau\, d\tau,$$

so that

$$\left.\begin{aligned} \frac{d\rho}{d\tau} &= \mu\Phi(\rho), \\ \frac{d\vartheta}{d\tau} &= \mu\Psi(\rho). \end{aligned}\right\} \tag{106}$$

Here

$$\left.\begin{aligned} \Phi(\rho) &= -\frac{1}{2\pi}\int_0^{2\pi} f(\rho\cos\xi, -\rho\sin\xi)\sin\xi\, d\xi, \\ \Psi(\rho) &= \frac{1}{2\pi\rho}\int_0^{2\pi} f(\rho\cos\xi, -\rho\sin\xi)\cos\xi\, d\xi \end{aligned}\right\} \tag{107}$$

where $\xi = \tau - \vartheta$.*

*One can make this substitution because the integrands are periodic functions.

Let us look at the first equation

$$\frac{d\rho}{d\tau} = \mu\Phi(\rho). \tag{108}$$

This is an equation of the first order and its state space degenerates to a line. A qualitative picture of an equation of this type is completely determined by the character and distribution of the equilibrium states on the state line. The coordinates of these equilibrium states are given by the roots of the equation

$$\Phi(\rho) = 0. \tag{109}$$

Let one of the roots of this equation be $\rho = \rho_0$. We shall examine the behavior of the representative point near this equilibrium state. We introduce a new variable u which measures the nearness of the representative point to the equilibrium state. In other words, we set

$$\rho = \rho_0 + u.$$

Then Equation (108) takes the form

$$\frac{du}{d\tau} = \mu\Phi(\rho_0 + u).$$

We expand the function $\Phi(\rho_0 + u)$ in a power series in u:

$$\Phi(\rho_0 + u) = \Phi'(\rho_0)\, u + \text{terms of higher order.}$$

If we limit ourselves to terms of the first degree in u, we get the equation of the first approximation

$$\frac{du}{d\tau} = \mu\Phi'(\rho_0)\, u.$$

A. M. Lyapunov has shown [2, 33], that when $\Phi'(\rho_0) \neq 0$, the equilibrium state is stable if $\Phi'(\rho_0) < 0$ and unstable if $\Phi'(\rho_0) > 0$. This can be qualitatively explained in the following manner.

For sufficiently small u one can neglect the terms of higher order; in other words, the sign of $\Phi'(\rho_0)$ will determine the sign of $du/d\tau$.*

Therefore the equilibrium state $\rho = \rho_0$ of Equation (108) is stable if

$$\Phi'(\rho_0) < 0, \tag{110}$$

and unstable if

$$\Phi'(\rho_0) > 0. \tag{111}$$

* We recall that $\mu > 0$.

One can find the equation of motion of the representative point on the state line by integrating Equation (108):

$$\mu(\tau - \tau_0) = \int_{\rho'}^{\rho} \frac{d\rho}{\Phi(\rho)},$$

where $\rho' = \rho_{\tau=\tau_0}$.

Let us look at the equation

$$\frac{d\vartheta}{d\tau} = \mu\Psi(\rho).$$

In practice it often happens that when

$$\Psi(\rho) \equiv 0,$$

it is also true that $d\vartheta/d\tau = 0$ and $\vartheta = \vartheta_0$ is a constant.

In this case all the integral curves on the ab-plane are lines passing through the origin at various constant angles of $\vartheta =$ constant. Motion along each of these lines is similar and is determined by Equation (108). The equilibrium states, which are determined by the equation $\Phi(\rho) = 0$, completely fill certain circles on the ab-plane, whose radii are equal to the roots of Equation (109). Figure 61 illustrates the ab-plane when Equation (109) has three roots, $\rho_1 = 0$, $\rho_2 < \rho_3$.

Let us transfer back to the xy-plane. By Equations (97), (98), and (105) we get

$$x = \rho \cos\vartheta \cos\tau + \rho \sin\vartheta \sin\tau = \rho \cos(\tau - \vartheta),$$

$$y = -\rho \cos\vartheta \sin\tau + \rho \sin\vartheta \cos\tau = -\rho \sin(\tau - \vartheta).$$

For any root $\rho = \rho_i$ of Equation (109) we have that

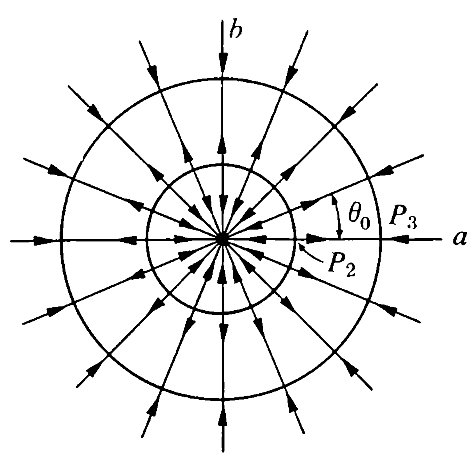


FIGURE 61

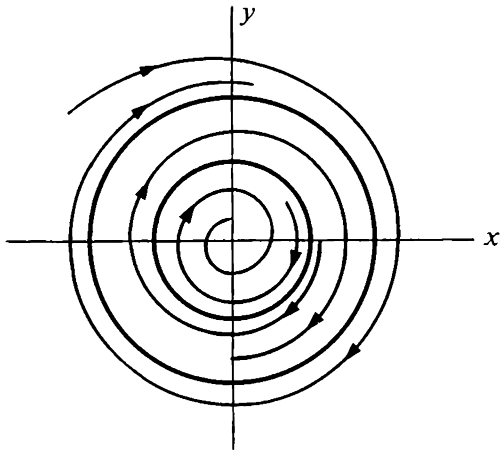


FIGURE 62

$$x = \rho_i \cos(\tau - \vartheta_0),$$
$$y = -\rho_i \sin(\tau - \vartheta_0), \tag{112}$$

that is, the limit cycle on the xy-plane is a circle.

The quantity ϑ_0 in the formulas of Equations (112) is arbitrary: this is in keeping with the fact that the equilibrium states are complete circles on the ab-plane.

If ρ_i corresponds to a stable equilibrium state, then the limit cycle on the xy-plane will be stable; all the neighboring integral curves are spirals which wind towards this cycle. If ρ_i corresponds to an unstable equilibrium state, then the limit cycle will be unstable on the xy-plane (Figure 62).

Let us look at the case $\Psi(\rho) \neq 0$. Then, for an arbitrary root $\rho = \rho_i$ of the equation $\Phi(\rho) = 0$ (we are assuming that the roots of the equation $\Phi(\rho) = 0$ do not coincide with the roots of the equation $\Psi(\rho) = 0$, since if they did, this case would revert to the one already examined) we have

$$\frac{d\vartheta}{d\tau} = \mu\Psi(\rho_i);$$

consequently,

$$\vartheta = \mu\Psi(\rho_i)\,\tau + \vartheta_0.$$

The motion of the representative point on the ab-plane takes place according to the equations

$$a = \rho_i \cos[\mu\Psi(\rho_i)\,\tau + \vartheta_0],$$
$$b = \rho_i \sin[\mu\Psi(\rho_i)\,\tau + \vartheta_0],$$

that is, along the limit cycle of radius $\rho = \rho_i$.

Whether this limit cycle is stable or unstable depends on the character of the equilibrium state at $\rho = \rho_i$. The direction of motion is determined by the sign of $\Psi(\rho_i)$. The other integral curves are spirals which wind away from or towards the limit cycle since ϑ is a function which does not change sign between the circles which have radii equal to the roots of the equation $\Psi(\rho) = 0$. Only integral curves which have radial tangents can intersect these circles.

Transferring back to the xy-plane, we get

$$x = \rho_i \cos \{[1 - \mu\Psi(\rho_i)]\tau - \vartheta_0\},$$

$$y = -\rho_i \sin \{[1 - \mu\Psi(\rho_i)]\tau - \vartheta_0\},$$

that is, we have circular limit cycles which correspond to the roots of the equation $\Phi(\rho) = 0$.

The difference from the case $\Psi(\rho) \equiv 0$ is that we get a correction in the frequency equal to $\Delta\omega = -\mu\Psi(\rho_i)$. The graph of the state plane remains the same.

In conclusion we note that the basis of the method of slowly varying parameters discussed here was developed by L. I. Mandel'shtam and N. D. Papaleksi [37].

By way of an example of an application of this method, we shall examine the so-called Froude pendulum [46].

To a shaft rotating with a uniform velocity Ω we attach a bearing (within which there is some frictional force) which is rigidly fastened to a pendulum (Figure 63). The equation for the motion of such a pendulum for small φ is

$$J\ddot{\varphi} + b\dot{\varphi} + Mgl\varphi = L, \tag{113}$$

where J is the moment of inertia of the pendulum relative to its axis of rotation, b is the coefficient of viscous resistance of the air, M is the mass of the pendulum, l is the distance from the axis of rotation to the center

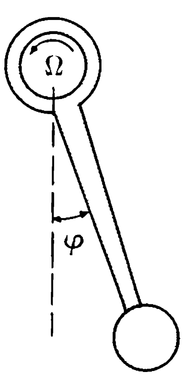


FIGURE 63

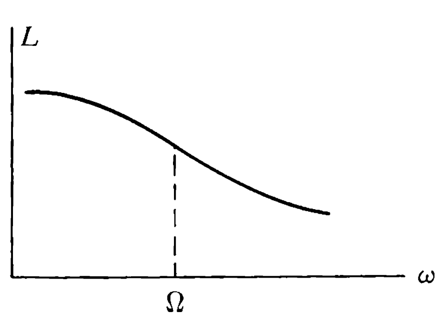


FIGURE 64

of gravity, and L is the torque due to friction between the shaft and the bearing.

Since the force of friction depends on the relative velocities of the moving surfaces, the torque due to friction will be a function of the relative angular velocity $\omega = \Omega - \dot{\varphi}$, that is,

$$L = L(\omega).$$

An example of what this function may be like is shown in Figure 64. As is obvious from the drawing, $L(\omega)$ can have intervals over which

$$L'(\omega) < 0.$$

If one chooses the velocity Ω so that when $\dot{\varphi} = 0$, $\Omega = \omega$ is a point of inflection in the curve $L(\Omega)$ in an interval where $L(\Omega)$ is decreasing, that is, $L''(\Omega) = 0$, $L'(\Omega) < 0$ then assuming that $\Omega \gg |\dot{\varphi}|$, and expanding the function $L(\omega) = L(\Omega - \dot{\varphi})$ in powers of $\dot{\varphi}$, we get

$$L(\Omega - \dot{\varphi}) = L(\Omega) - L'(\Omega)\dot{\varphi} - \tfrac{1}{6}L'''(\Omega)\dot{\varphi}^3 + \cdots.$$

If we restrict ourselves to these terms, we can rewrite the equation of motion of the pendulum as

$$\ddot{\varphi} + \frac{b}{J}\dot{\varphi} + \frac{Mgl}{J}\varphi = \frac{L(\Omega)}{J} - \frac{L'(\Omega)}{J}\dot{\varphi} - \frac{1}{6}\frac{L'''(\Omega)}{J}\dot{\varphi}^3.$$

Letting $\tau = kt$ where $(k = \sqrt{Mgl/J})$, we have

$$\frac{d^2\varphi}{d\tau^2} + \varphi = -\frac{b}{Jk}\frac{d\varphi}{d\tau} + \frac{L(\Omega)}{Jk^2} - \frac{L'(\Omega)}{Jk}\frac{d\varphi}{d\tau} - \frac{k}{6}\frac{L'''(\Omega)}{J}\left(\frac{d\varphi}{d\tau}\right)^3.$$

We introduce the new variable

$$\psi = \varphi - \frac{L(\Omega)}{Jk^2}.$$

The equation of motion of the pendulum near its equilibrium point takes the form

$$\frac{d^2\psi}{d\tau^2} + \psi = -\left(\frac{b}{Jk} + \frac{L'(\Omega)}{Jk}\right)\frac{d\psi}{d\tau} - \frac{k}{6}\frac{L'''(\Omega)}{J}\left(\frac{d\psi}{d\tau}\right)^3.$$

We suppose that this system is almost linear and conservative, that is, that the dimensionless quantities* satisfy

* We are assuming that the coefficient of viscosity is small and that the torque caused by Coulomb friction is nearly constant.

$$\frac{b}{Jk} \ll 1, \quad \frac{|L'(\Omega)|}{Jk} \ll 1, \quad \frac{k\,|L'''(\Omega)|}{J} \ll 1.$$

Let $\mu = b/Jk$ be a small parameter which characterizes how closely this system approximates a linear conservative one.

With the notation

$$\alpha = 1 + \frac{L'(\Omega)}{b}$$

and

$$\beta = \frac{k^2}{6}\frac{L'''(\Omega)}{b}$$

we can rewrite the equation of motion of the pendulum in the form

$$\frac{d^2\psi}{d\tau^2} + \psi = \mu\left[-\alpha\frac{d\psi}{d\tau} - \beta\left(\frac{d\psi}{d\tau}\right)^3\right]. \tag{114}$$

Equation (114) has the same form as Equation (94).

From Equation (107) we get

$$\Phi(\rho) = -\tfrac{1}{2}\rho\,(\alpha + \tfrac{3}{4}\beta\rho^2),$$
$$\Psi(\rho) \equiv 0.$$

The equilibrium states for the first equation of (106) are given by the roots of the equation

$$\rho\,(\alpha + \tfrac{3}{4}\beta\rho^2) = 0.$$

The roots of this equation are

$$\rho_1 = 0, \quad \rho_2 = \sqrt{-\frac{4}{3}\frac{\alpha}{\beta}}\,.$$

Obviously, the equilibrium state at $\rho = \rho_2$ will exist if α and β have opposite signs.

Whether an equilibrium state is stable or unstable depends on the signs of the derivative

$$\Phi'(\rho) = -\tfrac{1}{2}(\alpha + \tfrac{9}{4}\beta\rho^2)$$

where ρ corresponds to the equilibrium state.

Let us examine the several cases.

1. $\alpha < 0, \beta < 0.$

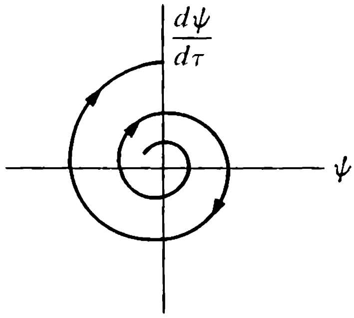


FIGURE 65

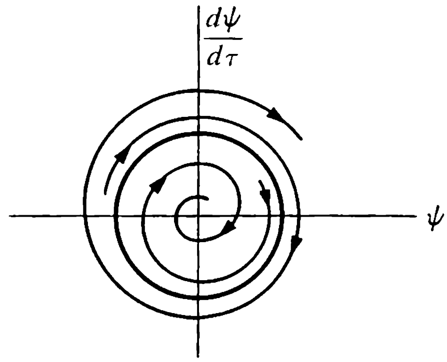


FIGURE 66

In this case there is only one equilibrium state at $\rho_1 = 0$, and $\Phi'(0) = -(\frac{1}{2})\,\alpha > 0$. Consequently, on the $\psi\,(d\psi/d\tau)$-plane there is a unique unstable equilibrium state (Figure 65), and the oscillations of the pendulum will steadily increase.

2. $\alpha > 0, \beta < 0$.

There will be two equilibrium states:

$$\rho_1 = 0, \quad \rho_2 = \sqrt{-\frac{4}{3}\frac{\alpha}{\beta}}.$$

For the equilibrium state at $\rho_1 = 0$, we have

$$\Phi'(0) = -\tfrac{1}{2}\alpha < 0,$$

which shows that this equilibrium state is stable.

When $\rho = \rho_2$,

$$\Phi'(\rho_2) = \alpha > 0,$$

that is, this equilibrium state is unstable and corresponds on the $\Psi\,(d\psi/d\tau)$-plane to an unstable limit cycle of radius $\rho = \rho_2$ (Figure 66). Consequently, for all $\rho < \rho_2$ the oscillations of the pendulum are damped and for all $\rho > \rho_2$ the amplitude of oscillations will increase steadily.

3. $\alpha > 0, \beta > 0$.

The unique equilibrium state at $\rho_1 = 0$ is stable and the oscillations of the pendulum are damped. The state plane is shown in Figure 67.

4. $\alpha < 0, \beta > 0$.

The equilibrium state at $\rho_1 = 0$ is unstable, but at $\rho = \rho_2$ it is stable. There is a stable limit cycle on the state plane (Figure 68). Hence the pendulum executes self-excited oscillations.* If $|\alpha|$ decreases, the limit cycle shrinks down to the origin. If $\alpha = 0$, the limit cycle merges with the

* Of course, the physical cause of these oscillations is external, the rotation of the suspension point of the pendulum.

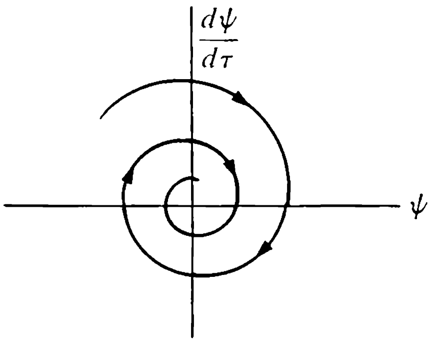


FIGURE 67

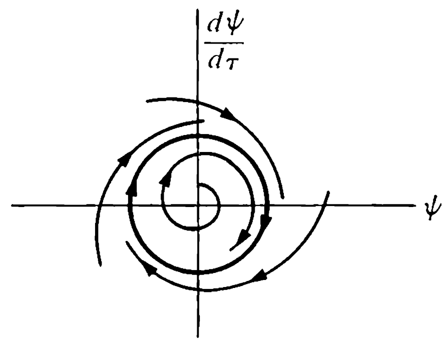


FIGURE 68

unstable equilibrium state at the origin and the origin becomes a stable point. Therefore, if α changes from a positive to a negative number, then after passing through the point $\alpha = 0$ there will be self-excited oscillations whose amplitudes increase continuously with $|\alpha|$. This mode of appearance of self-excited oscillations is called a "soft excitation."

The steady increasing oscillations which appear in the first two cases are impossible in practice since they require an unlimited external source of energy. This means that the approximation imposed by our expansion of L is inadequate and it is necessary to consider terms of higher order.

We put $L(\Omega - \dot{\varphi})$ in the form

$$L(\Omega - \dot{\varphi}) = L(\Omega) - L'(\Omega)\dot{\varphi} - \tfrac{1}{6}L'''(\Omega)\dot{\varphi}^3 - \tfrac{1}{120}L^{(V)}(\Omega)\dot{\varphi}^5.*$$

We keep the former assumptions and notations in force and introduce the letter

$$\gamma = \frac{k^4 L^{(V)}(\Omega)}{120b}.$$

Further, we suppose that

$$\frac{k^3|L^{(V)}(\Omega)|}{J} \ll 1.$$

As a result we get an equation of motion for the pendulum in the form

$$\frac{d^2\psi}{d\tau^2} + \psi = \mu\left[-\alpha\frac{d\psi}{d\tau} - \beta\left(\frac{d\psi}{d\tau}\right)^3 - \gamma\left(\frac{d\psi}{d\tau}\right)^5\right].$$

In this equation

$$\Phi(\rho) = -\tfrac{1}{2}\rho(\alpha + \tfrac{3}{4}\beta\rho^2 + \tfrac{5}{8}\gamma\rho^4),$$

* In the expansion of $L(\Omega - \dot{\varphi})$ we have not included the term of the fourth order because in the succeeding calculations (when we take the mean) it vanishes.

and

$$\Psi(\rho) \equiv 0.$$

We shall restrict our study to the case where $\beta < 0$ and $\gamma > 0$. Suppose that β and γ are fixed. One of the equilibrium states will always be at $\rho = 0$. We can find the other two* from the equation

$$\alpha + \tfrac{3}{4}\beta\rho^2 + \tfrac{5}{8}\gamma\rho^4 = 0. \tag{115}$$

On the $\alpha\rho^2$-plane Equation (115) describes a parabola which intersects the ρ^2-axis at the points

$$\alpha_1 = 0, \quad \rho_1^2 = 0$$

and

$$\alpha_2 = 0, \quad \rho_2^2 = -\frac{6}{5}\frac{\beta}{\gamma}.$$

The vertical tangent to this parabola is located at the point

$$\alpha = \frac{9}{40}\frac{\beta^2}{\gamma}, \quad \rho^2 = -\frac{3}{5}\frac{\beta}{\gamma}.$$

The shape of this parabola is shown in Figure 69.
The derivative of $\Phi(\rho)$ is

$$\Phi'(\rho) = -\tfrac{1}{2}[\alpha + \tfrac{9}{4}\beta\rho^2 + \tfrac{25}{8}\gamma\rho^4].$$

The parabola

$$\alpha + \tfrac{9}{4}\beta\rho^2 + \tfrac{25}{8}\gamma\rho^4 = 0 \tag{116}$$

on the $\alpha\rho^2$-plane divides the region where $\Phi'(\rho) > 0$ from the region where $\Phi'(\rho) < 0$. This parabola is shown by the dotted line in Figure 69. The region of instability lies inside the parabola (116) where $\Phi'(\rho) > 0$.

* We are interested only in the case $\rho > 0$.

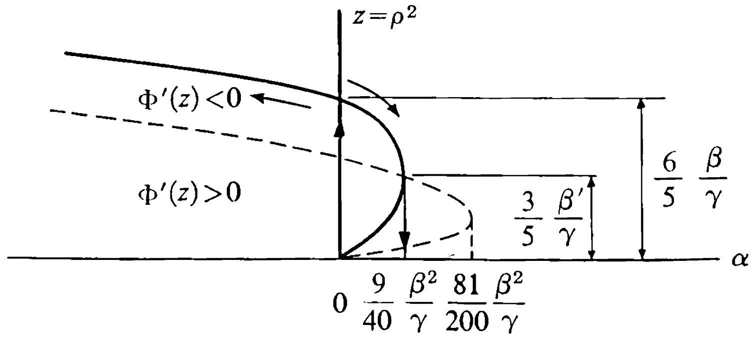


FIGURE 69

From Figure 69 we can see that when

$$\alpha > \alpha_0 = \frac{9}{40}\frac{\beta^2}{\gamma}$$

there will be only one stable equilibrium state. The form of the $(\psi, d\psi/d\tau)$-state plane for these values of α is shown in Figure 67. In this case the pendulum executes damped oscillations.

When $\alpha_0 > \alpha > 0$ there are three equilibrium states: a stable equilibrium state at $\rho = 0$, an unstable equilibrium state which corresponds to the lower branch of the parabola (115), a stable equilibrium state which corresponds to the upper branch of the parabola (115). On the $(\psi, d\psi/d\tau)$-state plane these equilibrium states correspond to a stable singular point at the origin, an unstable and a stable limit cycle as shown in Figure 70. This means that for all initial conditions inside the unstable limit cycle, the pendulum will have damped oscillations. And for all initial conditions lying outside this limit cycle, there will be stable self-excited oscillations.

Finally, if $\alpha < 0$, the equilibrium state at the origin is unstable and there is only one limit cycle, that is, for arbitrary initial conditions, the pendulum will have self-excited oscillations (Figure 71).

It is interesting to observe that when $\alpha_0 > \alpha > 0$ the pendulum may return either to rest or develop self-excited oscillations. If the pendulum is at rest, we can put it in a self-excited state by hitting it with a sufficiently large force.

Let us investigate how self-excited oscillations arise when α is varied from positive to negative values.

Suppose the pendulum is in its stable state at rest when $\alpha > 0$. If we let $\alpha = 0$, self-excited oscillations of finite amplitude will arise. As α

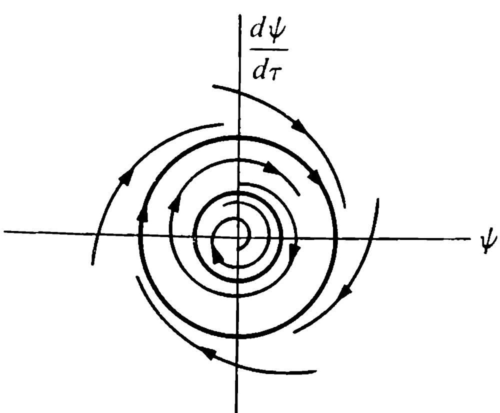


FIGURE 70

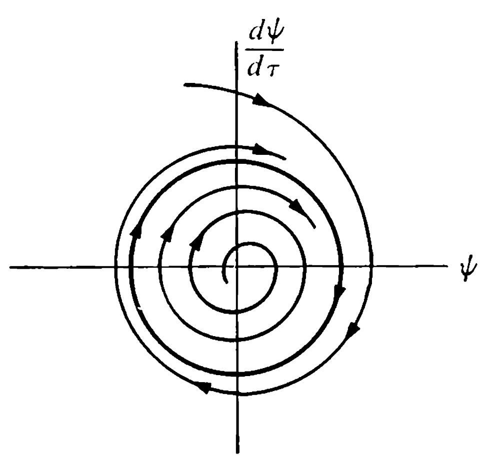


FIGURE 71

decreases the amplitude of the self-excited oscillations will gradually increase. This mode of appearance of self-excited oscillations is called "hard excitation."

In the case of the reverse variation in α from negative to positive values, the amplitude of the self-excited oscillations will gradually decrease and when $\alpha = \alpha_0$, they will abruptly cease and after a transient interval the system will reach its stable equilibrium state at rest.

This illustrates how the appearance and disappearance of self-excited oscillations occur for various values of the parameter α, and in this connection we are confronted with a phenomenon which is akin to hysteresis. The parameter α is frequently called "the disturbance coefficient."

8. The Method of Equivalent Linearization

The method of equivalent linearization and the methods of harmonic and energy balance, which are equivalent to it, have been developed and applied to a great number of nonlinear problems by N. M. Krylov and N. N. Bogolyubov [10], [28].

Important contributions have been made in the application of these methods to nonlinear problems in automatic control in the works of L. S. Goldfarb [21], V. A. Kotel'nik [27], R. J. Kochenburger [57], and especially E. P. Popov [42] who made important developments and applications in the investigation and analysis of nonlinear automatic systems. Since these methods have been very extensively described in the literature [42], we shall limit ourselves to a proof of the equivalence of the method of slowly changing coefficients and the method of equivalent linearization [10].

In the preceding section we showed that if one needs a solution of Equation (94), that is, the equation

$$\frac{d^2x}{d\tau^2} + x = \mu f\left(x, \frac{dx}{d\tau}\right), \tag{117}$$

in the form

$$x = \rho \cos(\tau - \vartheta), \tag{118}$$

then the shortened equations (the equations of the first approximation) have the form

$$\left.\begin{aligned} \frac{d\rho}{d\tau} &= \mu\Phi(\rho), \\ \frac{d\vartheta}{d\tau} &= \mu\Psi(\rho), \end{aligned}\right\} \tag{119}$$

where

$$\left.\begin{aligned}\Phi(\rho) &= -\frac{1}{2\pi}\int_0^{2\pi} f(\rho\cos\psi, -\rho\sin\psi)\sin\psi\, d\psi,\\ \Psi(\rho) &= \frac{1}{2\pi\rho}\int_0^{2\pi} f(\rho\cos\psi, -\rho\sin\psi)\cos\psi\, d\psi,\end{aligned}\right\} \tag{120}$$

$$\psi = \tau - \vartheta.$$

If we use the notation

$$\omega_e(\rho) = 1 - \mu\Psi(\rho), \tag{121}$$

then Equation (119) can be rewritten in the form

$$\left.\begin{aligned}\frac{d\rho}{d\tau} &= \mu\Phi(\rho),\\ \frac{d\psi}{d\tau} &= \omega_e(\rho).\end{aligned}\right\} \tag{122}$$

If we take the square of the expression (121) and neglect the terms containing μ^2 as a factor, then we get

$$\omega_e^2(\rho) = 1 - 2\mu\Psi(\rho) = 1 - \frac{\mu}{\pi\rho}\int_0^{2\pi} f(\rho\cos\psi, -\rho\sin\psi)\cos\psi\, d\psi.$$

By introducing the notation

$$h(\rho) = \frac{\mu}{2\pi\rho}\int_0^{2\pi} f(\rho\cos\psi, -\rho\sin\psi)\sin\psi\, d\psi = -\frac{\mu}{\rho}\Phi(\rho) \tag{123}$$

we can rewrite Equations (122):

$$\left.\begin{aligned}\frac{d\rho}{d\tau} &= -h(\rho)\,\rho,\\ \frac{d\psi}{d\tau} &= \omega_e(\rho).\end{aligned}\right\} \tag{124}$$

By differentiating the relation (118), that is,

$$x = \rho\cos\psi,$$

we get

$$\frac{dx}{d\tau} = \frac{d\rho}{d\tau}\cos\psi - \rho\frac{d\psi}{d\tau}\sin\psi$$

$$= -h(\rho)\,\rho\cos\psi - \rho\omega_e(\rho)\sin\psi. \tag{125}$$

By differentiating this expression once again, we get

$$\frac{d^2x}{d\tau^2} = -\frac{dh}{d\rho}\cdot\frac{d\rho}{d\tau}\,\rho\cos\psi - h\,\frac{d\rho}{d\tau}\cos\psi + h\rho\frac{d\psi}{d\tau}\sin\psi$$

$$-\frac{d\rho}{d\tau}\omega_e\sin\psi - \rho\frac{d\omega_e}{d\rho}\cdot\frac{d\rho}{d\tau}\sin\psi - \rho\omega_e\frac{d\psi}{d\tau}\cos\psi.$$

Now, making use of Equations (124) and (125), we write

$$\frac{d^2x}{d\tau^2} + 2h(\rho)\frac{dx}{d\tau} + \omega_e^2(\rho)\,x = Q(\mu^2),$$

where $Q(\mu^2)$ has the same order as μ^2 in a neighborhood of zero.

Therefore, one can assert that a solution to Equation (118) with an accuracy up to a quantity of order μ^2 in a neighborhood of zero will satisfy the linear differential equation of the form

$$\frac{d^2x}{d\tau^2} + 2h(\rho)\frac{dx}{d\tau} + \omega_e^2(\rho)\,x = 0. \tag{126}$$

Consequently, the first approximation of the oscillator described by Equation (117) is equivalent (up to a quantity of order μ^2) to some linear oscillator described by Equation (126). This linear oscillator is called the equivalent system [10].

We note that Equation (126) can be obtained from Equation (117) in the following manner.

By making the substitution $x = \rho\cos\psi$ and $dx/d\tau = -\rho\sin\psi$ in the nonlinear term of Equation (117) and expanding it in a Fourier series, we get

$$\mu f(\rho\cos\psi, -\rho\sin\psi) = \frac{a_0}{2} + a_1\cos\psi + b_1\sin\psi + \cdots.$$

If we restrict ourselves to the first three terms and suppose that the nonlinear function has the property that the free term a_0 is equal to zero, we get

$$\mu f(\rho\cos\psi, -\rho\sin\psi) = \frac{a_1}{\rho}x - \frac{b_1}{\rho}\frac{dx}{d\tau} \tag{127}$$

$$\left(\cos\psi = \frac{x}{\rho},\ \sin\psi = -\frac{dx}{d\tau}\cdot\frac{1}{\rho}\right)$$

where

$$\left.\begin{aligned} a_1 &= \frac{\mu}{\pi}\int_0^{2\pi} f(\rho\cos\psi, -\rho\sin\psi)\cos\psi\, d\psi, \\ b_1 &= \frac{\mu}{\pi}\int_0^{2\pi} f(\sigma\cos\psi, -\sigma\sin\psi)\sin\psi\, d\psi. \end{aligned}\right\} \tag{128}$$

Substituting (127) into (117), we can rewrite the latter in the form

$$\frac{d^2x}{d\tau^2} + \frac{b_1}{\rho}\frac{dx}{d\tau} + \left(1 - \frac{a_1}{\rho}\right)x = 0.$$

If we note that

$$h(\rho) = \frac{\mu}{2\pi\rho}\int_0^{2\pi} f(\sigma\cos\psi, -\sigma\sin\psi)\sin\psi\, d\psi = \frac{b_1}{2\rho},$$

$$\omega_e^2(\rho) = 1 - \frac{\mu}{\pi\rho}\int_0^{2\pi} f(\rho\cos\psi, -\rho\sin\psi)\cos\psi\, d\psi = 1 - \frac{a_1}{\rho},$$

we have finally

$$\frac{d^2x}{d\tau^2} + 2h(\rho)\frac{dx}{d\tau} + \omega_e^2(\rho)x = 0.$$

In conclusion, we note that the method of harmonic (equivalent) linearization has been worked out by E. P. Popov both for symmetric and nonsymmetric linear systems, and also for linear systems of higher orders.

II

Oscillations in Nonlinear Autonomous Systems with Two Degrees of Freedom

In this chapter we will study nonlinear systems with two degrees of freedom, which are almost linear conservative systems, [12], [13], [34].

Specific peculiarities of these systems are illustrated with concrete examples, and conditions for the existence of periodic and biharmonic motions are found.

The method of slowly varying parameters is used in this investigation.

9. Systems Without Gyroscopic Forces

First we shall look at the way in which the method of slowly varying parameters is applied to a system with two degrees of freedom.

Let the motion of the dynamic system be described by the following system of differential equations

$$\left.\begin{aligned} \ddot{x} + A_1\ddot{y} + B_1 y + n_1^2 x &= \mu f(x,\dot{x},y,\dot{y}), \\ \ddot{y} + A_2\ddot{x} + B_2 x + n_2^2 y &= \mu g(x,\dot{x},y,\dot{y}), \end{aligned}\right\} \qquad (129)$$

where A_1, A_2, B_1, B_2, n_1^2, n_2^2 are constant coefficients, μ is a small parameter which measures the closeness of the approximation of the given system to a linear conservative one, and f and g are nonlinear functions.

For $\mu = 0$ the solution of the system is

$$\left.\begin{aligned} x &= a \sin(k_1 t + \beta_1) + b \sin(k_2 t + \beta_2), \\ y &= \alpha_1 a \sin(k_1 t + \beta_1) + \alpha_2 b \sin(k_2 t + \beta_2), \end{aligned}\right\} \qquad (130)$$

where a, b, β_1, and β_2 are arbitrary constants of integrations, k_1 and k_2 (we choose $k_2 > k_1$) are the characteristic (normal) frequencies which are the roots of the equation

$$\sigma k^4 - (n_1^2 + n_2^2 - A_1B_2 - A_2B_1)\,k^2 + n_1^2n_2^2 - B_1B_2 = 0,^* \quad (\sigma = 1 - A_1A_2), \tag{131}$$

and where α_1 and α_2 are given by the formulas

$$\left.\begin{aligned} \alpha_1 &= \frac{A_2k_1^2 - B_2}{n_2^2 - k_1^2} = \frac{n_1^2 - k_1^2}{A_1k_1^2 - B_1}, \\ \alpha_2 &= \frac{A_2k_2^2 - B_2}{n_2^2 - k_2^2} = \frac{n_1^2 - k_2^2}{A_1k_2^2 - B_1}. \end{aligned}\right\} \tag{132}$$

Now, we want to find a solution of the system (129) when $\mu \neq 0$ in the same form as (130) but where a, b, β_1, and β_2 are functions of time.

In a manner analogous to the case of a system with one degree of freedom, we impose on the functions a, b, β_1 and β_2 conditions which permit us to take the derivatives of x and y as if a, b, β_1, and β_2 were constants. Consequently, differentiating x and y with respect to time, we get

$$\begin{aligned} \dot{x} &= ak_1 \cos(k_1t + \beta_1) + bk_2 \cos(k_2t + \beta_2), \\ \dot{y} &= \alpha_1ak_1 \cos(k_1t + \beta_1) + \alpha_2bk_2 \cos(k_2t + \beta_2) \end{aligned} \tag{133}$$

and

$$\dot{a} \sin(k_1t + \beta_1) + \dot{b} \sin(k_2t + \beta_2) + a\dot{\beta}_1 \cos(k_1t + \beta_1) + b\dot{\beta}_2 \cos(k_2t + \beta_2) = 0; \tag{134}$$

$$\alpha_1\dot{a} \sin(k_1t + \beta_1) + \alpha_2\dot{b} \sin(k_2t + \beta_2) + \alpha_1a\dot{\beta}_1 \cos(k_1t + \beta_1) + \alpha_2b\dot{\beta}_2 \cos(k_2t + \beta_2) = 0. \tag{135}$$

Now, by taking the second derivatives of x and y with respect to time (that is, differentiating the expression (133) with respect to time) and substituting these derivatives into Equation (129), we have (after taking (132) into account)

$$\begin{aligned} \mu f^* = {} & \dot{a}k_1 (1 + \alpha_1A_1) \cos(k_1t + \beta_1) + \dot{b}k_2 (1 + \alpha_2A_1) \cos(k_2t + \beta_2) \\ & - k_1a\dot{\beta}_1 (1 + \alpha_1A_1) \sin(k_1t + \beta_1) - k_2b\dot{\beta}_2 (1 + \alpha_2A_1) \sin(k_2t + \beta_2), \end{aligned} \tag{136}$$

and

$$\mu g^* = \dot{a}k_1(\alpha_1 + A_2)\cos(k_1t + \beta_1) + bk_2(\alpha_2 + A_2)\cos(k_2t + \beta_2) - k_1a\dot{\beta}_1(\alpha_1 + A_2)\sin(k_1t + \beta_1) - k_2b\dot{\beta}_2(\alpha_2 + A_2)\sin(k_2t + \beta_2), \tag{137}$$

where

$$\left.\begin{aligned}
f^* &= f[a\sin(k_1t + \beta_1) + b\sin(k_2t + \beta_2),\\
&\quad ak_1\cos(k_1t + \beta_1) + bk_2\cos(k_2t + \beta_2),\\
&\quad \alpha_1a\sin(k_1t + \beta_1) + \alpha_2b\sin(k_2t + \beta_2),\\
&\quad \alpha_1ak_1\cos(k_1t + \beta_1) + \alpha_2bk_2\cos(k_2t + \beta_2)],\\
g^* &= g[a\sin(k_1t + \beta_1) + b\sin(k_2t + \beta_2),\\
&\quad ak_1\cos(k_1t + \beta_1) + bk_2\cos(k_2t + \beta_2),\\
&\quad \alpha_1a\sin(k_1t + \beta_1) + \alpha_2b\sin(k_2t + \beta_2),\\
&\quad \alpha_1ak_1\cos(k_1t + \beta_1) + \alpha_2bk_2\cos(k_2t + \beta_2)].
\end{aligned}\right\} \tag{138}$$

Equations (134) to (137) are a system of equations in the variables $\dot{a}$, $\dot{b}$, $a\dot{\beta}_1$, and $b\dot{\beta}_2$. The determinant of this system is

$$\Delta = \begin{vmatrix} \sin\xi, & \sin\eta, & \cos\xi, & \cos\eta \\ \alpha_1\sin\xi, & \alpha_2\sin\eta, & \alpha_1\cos\xi, & \alpha_2\cos\eta \\ k_1D_1\cos\xi, & k_2D_2\cos\eta, & -k_1D_1\sin\xi, & -k_2D_2\sin\eta \\ k_1D_3\cos\xi, & k_2D_4\cos\eta, & -k_1D_3\sin\xi, & -k_2D_4\sin\eta \end{vmatrix}$$

$$= k_1k_2\sigma(\alpha_2 - \alpha_1)^2,$$

where

$$D_1 = 1 + \alpha_1A_1,\ D_2 = 1 + \alpha_2A_1,\ D_3 = \alpha_1 + A_2,\ D_4 = \alpha_2 + A_2,$$

$$\xi = k_1t + \beta_1,\ \eta = k_2t + \beta_2.$$

After substituting for the elements of the first column the right sides of Equations (134–137) we get

$$\Delta = \begin{vmatrix} 0, & \sin\eta, & \cos\xi, & \cos\eta \\ 0, & \alpha_2\sin\eta, & \alpha_1\cos\xi, & \alpha_2\cos\eta \\ \mu f^*, & k_2D_2\cos\eta, & -k_1D_1\sin\xi, & -k_2D_2\sin\eta \\ \mu g^*, & k_2D_4\cos\eta, & -k_1D_3\sin\xi, & -k_2D_4\sin\eta \end{vmatrix}$$

$$= \mu(\alpha_2 - \alpha_1)k_2[(\alpha_2 + A_2)f^* - (1 + \alpha_2A_1)g^*]\cos\xi.$$

Therefore, we have

$$\frac{da}{dt} = \frac{\Delta_1}{\Delta} = \frac{\mu}{k_1\sigma(\alpha_2 - \alpha_1)}[(\alpha_2 + A_2)f^* - (1 + \alpha_2 A_1)\, g^*]\cos\xi. \quad (139)$$

From (132) we get

$$\alpha_2 - \alpha_1 = \frac{(k_2^2 - k_1^2)(A_2 n_2^2 - B_2)}{(n_2^2 - k_1^2)(n_2^2 - k_2^2)}, \quad \alpha_2 + A_2 = \frac{A_2 n_2^2 - B_2}{n_2^2 - k_2^2},$$

$$1 + \alpha_2 A_1 = \frac{n_2^2 - A_1 B_2 - \sigma k_2^2}{n_2^2 - k_2^2}.$$

Since from (131) we have

$$\sigma(k_1^2 + k_2^2) = n_1^2 + n_2^2 - A_1 B_2 - A_2 B_1,$$

it follows that

$$1 + \alpha_2 A_1 = -\frac{n_1^2 - k_1^2}{n_2^2 - k_2^2}\left(1 + \frac{A_2}{\alpha_1}\right) = -\frac{n_1^2 - k_1^2}{n_2^2 - k_2^2}\cdot\frac{A_2 n_2^2 - B_2}{A_2 k_1^2 - B_2}.$$

By substituting all these expressions in (139) and taking (132) into account we get

$$\frac{da}{dt} = \frac{\mu}{k_1\sigma(k_2^2 - k_1^2)}\left[\frac{A_2 k_1^2 - B_2}{\alpha_1} f^* + (A_1 k_1^2 - B_1)\, g^*\right]\cos\xi. \quad (140)$$

Analogously, we have

$$\frac{db}{dt} = -\frac{\mu}{k_2\sigma(k_2^2 - k_1^2)}\left[\frac{A_2 k_2^2 - B_2}{\alpha_2} f^* + (A_1 k_2^2 - B_1)\, g^*\right]\cos\eta, \quad (141)$$

$$a\frac{d\beta_1}{dt} = -\frac{\mu}{k_1\sigma(k_2^2 - k_1^2)}\left[\frac{A_2 k_1^2 - B_2}{\alpha_1} f^* + (A_1 k_1^2 - B_1)\, g^*\right]\sin\xi, \quad (142)$$

and

$$b\frac{d\beta_2}{dt} = \frac{\mu}{k_2\sigma(k_2^2 - k_1^2)}\left[\frac{A_2 k_2^2 - B_2}{\alpha_2} f^* + (A_1 k_2^2 - B_1)\, g^*\right]\sin\eta. \quad (143)$$

The resulting expressions give a transformation of the system (129) to new variables. From these equations it follows that for sufficiently small μ the derivatives $da/d\tau$, $db/d\tau$, $d\beta_1/d\tau$, and $d\beta_2/d\tau$ are also small. Consequently, a, b, β_1, and β_2 are slowly changing functions of time. We are assuming that the variation in these quantities is relatively small compared to the waves in the resulting dynamic system. Thus, by averaging the right sides of these equations over the periods $2\pi/k_1$ and $2\pi/k_2$, we get approximate equations for the determination of a, b, β_1, and β_2:

$$
\left.\begin{aligned}
\frac{da}{dt} &= \frac{\mu}{2k_1\sigma(k_2^2 - k_1^2)}\left[\frac{A_2k_1^2 - B_2}{\alpha_1}F_1 + (A_1k_1^2 - B_1)\,G_1\right],\\
\frac{db}{dt} &= -\frac{\mu}{2k_1\sigma(k_2^2 - k_1^2)}\left[\frac{A_2k_2^2 - B_2}{\alpha_2}F_2 + (A_1k_2^2 - B_1)\,G_2\right],
\end{aligned}\right\} \tag{144}
$$

$$
\left.\begin{aligned}
\frac{d\beta_1}{dt} &= -\frac{\mu}{2ak_1\sigma(k_2^2 - k_1^2)}\left[\frac{A_2k_1^2 - B_2}{\alpha_1}F_3 + (A_1k_1^2 - B_1)\,G_3\right],\\
\frac{d\beta_2}{dt} &= \frac{\mu}{2bk_2\sigma(k_2^2 - k_1^2)}\left[\frac{A_2k_2^2 - B_2}{\alpha_2}F_4 + (A_1k_2^2 - B_2)\,G_4\right],
\end{aligned}\right\} \tag{145}
$$

where

$$
\left.\begin{aligned}
F_1 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\cos\xi\,d\xi\,d\eta, & G_1 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\cos\xi\,d\xi\,d\eta;\\
F_2 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\cos\eta\,d\xi\,d\eta, & G_2 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\cos\eta\,d\xi\,d\eta;\\
F_3 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\sin\xi\,d\xi\,d\eta, & G_3 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\sin\xi\,d\xi\,d\eta;\\
F_4 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\sin\eta\,d\xi\,d\eta, & G_4 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\sin\eta\,d\xi\,d\eta.
\end{aligned}\right\} \tag{146†}
$$

Here $\xi = k_1t + \beta_1$, $\eta = k_2t + \beta_2$, and f^* and g^* are defined by Equations (138).

Equations (144) and (145) can be found in another way. With this in mind we look for a solution of (129) in the form

$$
\left.\begin{aligned}
x &= a\sin(k_1t + \beta_1) + b\sin(k_2t + \beta_2),\\
y &= \alpha_1a\sin(k_1t + \beta_1) + \alpha_2b\sin(k_2t + \beta_2)\\
&\quad + a_1\cos(k_1t + \beta_1 + \gamma_1) + b\cos(k_2t + \beta_2 + \gamma_2),
\end{aligned}\right\} \tag{147}
$$

---

*We are considering the case where the roots of this equation are distinct and nonzero.

†In computing these integrals the quantities a, b, β_1, β_2 are considered constant since their variation in one period is considered negligible.

where a, b, β_1, β_2 are slowly changing functions of time and their first derivatives with respect to time have the order of μ, their second derivatives have the order of μ^2, and so on.

a_1, b_1, γ_1, γ_2 are slowly changing functions of time which are of the order of μ. Their first derivatives are of the order of μ^2, the second derivatives of the order of μ^3, and so on.

If we substitute (147) in Equations (129) and neglect all terms which are of the order of μ^2, then, after having taken the averages over the periods $2\pi/k_1$ and $2\pi/k_2$, we arrive at the Equations (144) and (145).

The right sides of Equations (144) and (145) do not depend on β_1 and β_2. Therefore the equations of (144) will serve to investigate the amplitudes a and b. The values of a and b which correspond to stationary states are found from the equations which we get by setting the right parts of Equations (144) equal to zero:

$$\frac{A_2k_1^2 - B_2}{\alpha_1} F_1 + (A_1k_1^2 - B_1)\, G_1 = 0,$$

$$\frac{A_2k_2^2 - B_2}{\alpha_2} F_2 + (A_1k_2^2 - B_1)\, G_2 = 0.$$

The values of a and b determined by these equations are substituted in Equation (145), and after this the corrections in the frequencies over a stationary state can be determined.

Remark. It is clear that this method can be applied to a system with n degrees of freedom

$$\sum_{s=1}^{n} (a_{is}\ddot{q}_s + c_{is}q_s) = \mu f_i\,(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$
$$(i = 1, 2, \ldots, n),$$

where a_{is}, c_{is} are constant coefficients and f_i are nonlinear functions.

In this case it is necessary to take averages over n periods.

To illustrate this general method we shall examine a concrete example of a nonlinear mechanical system.

Kelvin's Problem. In a work which he called "On The Dependence of the Motion of a Clock on its Means of Suspension," [61] Lord Kelvin examined the affect of the means of suspension on the motion of a clock frame and its balance wheel. He based his discussion on the linear theory of oscillations, and succeeded in calculating corrections to the motion of a clock constructed with various techniques of suspension. He also obtained graphs of these phenomena for the suspension of clocks with two degrees of freedom. However, Lord Kelvin's study did not give a complete

dynamic explanation of the phenomenon because his principles for deriving the equations of oscillations were experimental.

In view of the suspension of the clock frame we assume that there is a torque proportional to the angle y of rotation about the vertical axis. In addition to a torque which is proportional to the angle x of rotation of the balance wheel relative to the clock frame, there is also a disturbance mechanism which acts on the balance wheel in such a way that each time it passes through the equilibrium position at $x = 0$ an impulse I_0 is imparted to it.

We adopt the following notation:

J_1 is the moment of inertia of the balance wheel relative to a vertical axis passing through its center.

m is the mass of the balance wheel.

h is the distance between the axis of rotation of the clock frame and the axis of rotation of the balance wheel.

J_2 is the moment of inertia of the clock frame relative to its axis of rotation.

c_1 and c_2 are coefficients of proportionality in the expressions for the restoring torques acting on the clock frame and the balance wheel.

b_1 and b_2 are coefficients of proportionality in the expressions for the torques caused by viscous friction.

We introduce into our considerations the Dirac delta function, which is defined in the following manner: $\delta(x) = 0$ when $x \neq 0$ and

$$\int_{-0}^{+0} \delta(x)\,dx = 1.$$

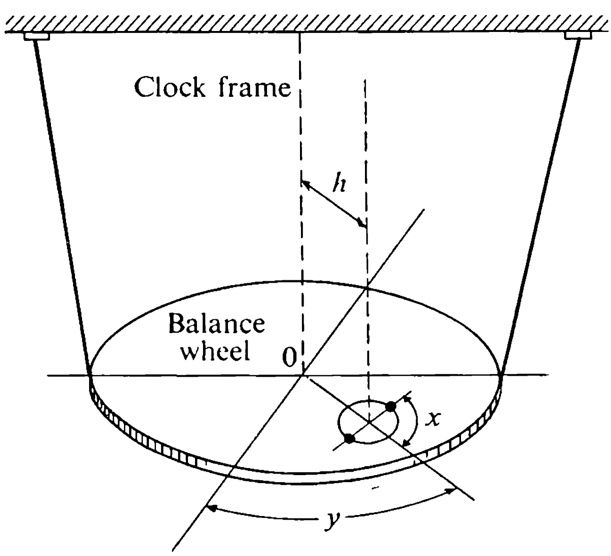


FIGURE 72

The expression for the kinetic energy of the system is given by

$$T = \tfrac{1}{2}(J_1 + J_2 + mh^2)\,\dot{y}^2 + \tfrac{1}{2}J_1(\dot{x}^2 + 2\dot{x}\dot{y}).$$

The potential energy due to the restoring torques is

$$\Pi = \frac{c_1 x^2}{2} + \frac{c_2 y^2}{2}.$$

Consequently, the equations for the motion of the system are written in the form

$$\ddot{x} + A_1\ddot{y} + n_1^2 x = n_1^2\left[-\frac{b_1}{c_1}\dot{x} + \frac{I_0}{c_1}\dot{x}\delta(x)\right],$$

$$\ddot{y} + A_2\ddot{x} + n_2^2 y = n_2^2\left[-\frac{b_2}{c_2}\dot{y}\right],$$

where

$$A_1 = 1, \quad A_2 = \frac{J_1}{J_1 + J_2 + mh^2}, \quad n_1^2 = \frac{c_1}{J_1}, \quad n_2^2 = \frac{c_2}{J_1 + J_2 + mh^2}.$$

We shall formulate the assumptions which characterize the nearness of this system to a linear conservative one.

1. We suppose that the damping in both parts of the system is small, that is,

$$\frac{b_1}{c_1}n_1 \ll 1, \quad \frac{b_2}{c_2}n_2 \ll 1.$$

2. We suppose that the impulses are "small"

$$\frac{I_0 n_1}{c_1} \ll 1.$$

By introducing the dimensionless parameter

$$\mu = \frac{b_1}{c_1}n_1,$$

which characterizes the nearness of this system to a linear conservative one, we can write the equations for the motion in the form

$$\left.\begin{aligned} \ddot{x} + A_1\ddot{y} + n_1^2 x &= \mu\frac{n_1^2}{n_1}[-\dot{x} + I\dot{x}\delta(x)], \\ \ddot{y} + A_2\ddot{x} + n_2^2 y &= \mu\frac{n_2^2}{n_1}[-\gamma\dot{y}], \end{aligned}\right\} \tag{148}$$

where

$$I = \frac{I_0}{b_1},$$

$$\gamma = \frac{b_2 c_1}{b_1 c_2}.$$

The expressions (148) have the same form as Equations (129) and, consequently, if we are looking for a solution in the form

$$\left.\begin{aligned} x &= a \sin (k_1 t + \beta_1) + b \sin (k_2 t + \beta_2), \\ y &= \alpha_1 a \sin (k_1 t + \beta_1) + \alpha_2 b \sin (k_2 t + \beta_2), \end{aligned}\right\} \tag{149}$$

then k_1 and k_2 will be the roots of the equation

$$\sigma k^4 - (n_1^2 + n_2^2) k^2 + n_1^2 n_2^2 = 0$$

$$(\sigma = 1 - A_1 A_2),$$

since

$$B_1 = B_2 = 0.$$

The coefficients α_1, α_2 are determined by the formulas

$$\alpha_1 = \frac{A_2 k_1^2}{n_2^2 - k_1^2} = \frac{n_1^2 - k_1^2}{A_1 k_1^2},$$

$$\alpha_2 = \frac{A_2 k_2^2}{n_2^2 - k_2^2} = \frac{n_1^2 - k_2^2}{A_1 k_2^2}.$$

Equations (144) and (145) have the form

$$\left.\begin{aligned} \frac{da}{d\tau} &= \frac{k_1}{n_1^3} \left[A_1 G_1 + \frac{A_2}{\alpha_1} F_1 \right], \\ \frac{db}{d\tau} &= -\frac{k_2}{n_1^3} \left[A_1 G_2 + \frac{A_2}{\alpha_2} F_2 \right], \end{aligned}\right\} \tag{150}$$

$$\left.\begin{aligned} \frac{d\beta_1}{d\tau} &= -\frac{k_1}{a n_1^3} \left[A_1 G_3 + \frac{A_2}{\alpha_1} F_3 \right], \\ \frac{d\beta_2}{d\tau} &= \frac{k_2}{b n_1^3} \left[A_1 G_4 + \frac{A_2}{\alpha_2} F_4 \right], \end{aligned}\right\} \tag{151}$$

where $\tau = \mu n_1^3 t / 2\sigma (k_2^2 - k_1^2)$ and F_1, F_2, F_3, F_4, G_1, G_2, G_3, and G_4 can be

found from the formulas of (146). Since in this case

$$f(x,x,y,\dot{y}) = \frac{n_1^2}{n_1}[-x + I\delta(x)\dot{x}],$$

$$g(x,\dot{x},y,\dot{y}) = \frac{n_2^2}{n_1}[-\gamma\dot{y}],$$

we have

$$F_1 = \frac{n_1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}[-k_1 a\cos\xi - k_2 b\cos\eta + I\dot{x}\delta(x)]\cos\xi\, d\xi\, d\eta, \quad (152)$$

where

$$x = a\sin\xi + b\sin\eta,\ \ \dot{x} = ak_1\cos\xi + bk_2\cos\eta.$$

The expression (152) can be rewritten in the form

$$F_1 = -n_1k_1a + \frac{In_1}{2\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi}\frac{d\xi}{dt}\cos\xi\,\delta(x)\,dx$$

$$= -n_1k_1a + \frac{In_1k_1}{2\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi}\cos\xi\,\delta(x)\,dx,$$

$$\left(\frac{d\xi}{dt} = k_1\right).$$

We shall compute the integral

$$\int_0^{2\pi}\cos\xi\,\delta(x)\,dx,$$

and suppose that a and b are positive. Let $a > b$, and then

$$\int_0^{2\pi}\cos\xi\,\delta(x)\,dx = \int_{+0}^{-0}\cos\xi_1\delta(x)\,dx + \int_{-0}^{+0}\cos\xi_2\delta(x)\,dx,^*$$

where

$$\cos\xi_1 = -\sqrt{1 - \left(\frac{b}{a}\right)^2\sin^2\eta},$$

*For $\xi = \xi_1$ and $\xi = \xi_2$ the value of x is zero.

$$\cos \xi_2 = \sqrt{1 - \left(\frac{b}{a}\right)^2 \sin^2 \eta}.$$

In this way we have

$$F_1 = -n_1 k_1 a + \frac{4 I n_1 k_1}{\pi^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{b}{a}\right)^2 \sin^2 \eta \, d\eta}. \tag{153}$$

Now, let $a < b$. In this case we have

$$F_1 = -n_1 k_1 a + \frac{4 I n_1 k_1}{\pi^2} \int_0^{\eta_1} \sqrt{1 - \left(\frac{b}{a}\right)^2 \sin^2 \eta \, d\eta}, \tag{154}$$

where $\eta_1 = \arc\sin a/b$.

Graphs of the function $\xi + b \sin \eta = 0$ for $a > b$ and $a < b$, are shown in Figures 73 and 74, and an examination of them will clarify the process of integration.

We shall transform the integral

$$\Phi = \int_0^{\eta_1} \sqrt{1 - \left(\frac{b}{a}\right)^2 \sin^2 \eta \, d\eta}.$$

To this purpose, we make the change of variable

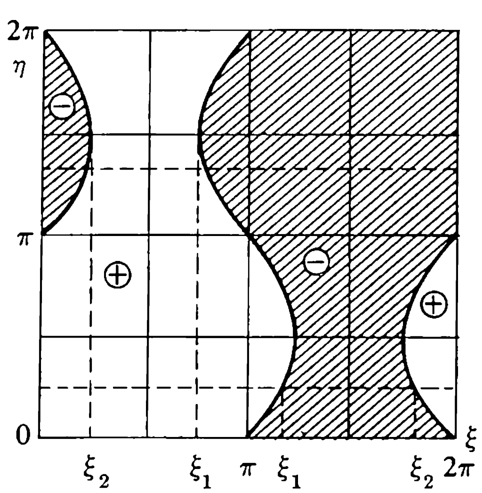


FIGURE 73

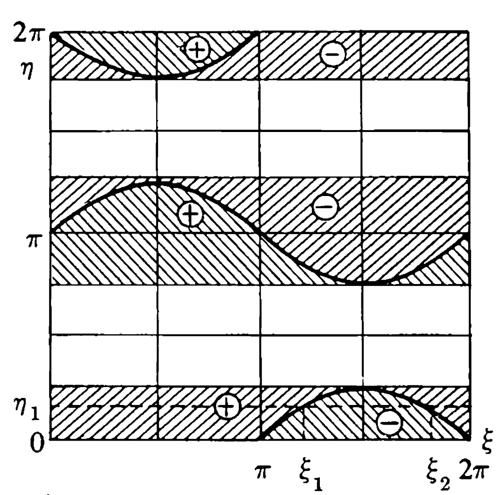


FIGURE 74

$$\frac{b}{a}\sin\eta = \sin\psi.$$

Then

$$\Phi = \frac{a}{b}\int_0^{\frac{\pi}{2}} \frac{\cos^2\psi\, d\psi}{\sqrt{1-\left(\frac{a}{b}\right)^2\sin^2\psi}}$$

$$= \frac{a}{b}\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-\left(\frac{a}{b}\right)^2\sin^2\psi}} + \frac{b}{a}\int_0^{\frac{\pi}{2}} \frac{-\left(\frac{a}{b}\right)^2\sin^2\psi\, d\psi}{\sqrt{1-\left(\frac{a}{b}\right)^2\sin^2\psi}}$$

$$= \left[\left(\frac{a}{b}\right) - \left(\frac{b}{a}\right)\right] K\left(\frac{a}{b}\right) + \frac{b}{a} E\left(\frac{a}{b}\right),$$

where

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-m^2\sin^2\psi}},$$

$$E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1-m^2\sin^2\psi}\, d\psi$$

are elliptic integrals of the first and second kind.

Therefore, we have that for $a > b$

$$F_1 = -k_1 n_1 a + \frac{4In_1k_1}{\pi^2} E\left(\frac{b}{a}\right),$$

and for $a < b$,

$$F_1 = -k_1 n_1 a + \frac{4In_1k_1}{\pi^2}\left\{\left[\left(\frac{a}{b}\right) - \left(\frac{b}{a}\right)\right] K\left(\frac{a}{b}\right) + \frac{b}{a} E\left(\frac{a}{b}\right)\right\}.$$

Analogously, we get that for $a > b$, $a > 0$, $b > 0$,

$$F_2 = -n_1 b k_2 + \frac{4In_1k_2}{\pi^2}\left\{\left[\left(\frac{b}{a}\right) - \left(\frac{a}{b}\right)\right] K\left(\frac{b}{a}\right) + \left(\frac{a}{b}\right) E\left(\frac{b}{a}\right)\right\}$$

and for $a < b$,

$$F_2 = -n_1 b k_2 + \frac{4In_1k_2}{\pi^2} E\left(\frac{a}{b}\right).$$

And, correspondingly, for G_1 and G_2 we get

$$G_1 = -\frac{n_2^2}{n_1}\gamma\alpha_1 k_1 a, \quad G_2 = -\frac{n_2^2}{n_1}\gamma\alpha_2 k_2 b.$$

From these calculations we see that F_3, F_4, G_3, and G_4 are equal to zero.

By introducing the variables $u = a\pi^2/4I$ and $v = b\pi^2/4I$, we can rewrite the Equations (150) for this problem in the form

$$\left.\begin{aligned} \frac{du}{d\tau} &= M\left[E\left(\frac{v}{u}\right) - \alpha u\right] = P_1(u,v), \\ \frac{dv}{d\tau} &= N\frac{v}{u}\left\{\left[1 - \left(\frac{u}{v}\right)^2\right] K\left(\frac{v}{u}\right) + \left(\frac{u}{v}\right)^2 E\left(\frac{v}{u}\right) - \beta u\right\} = Q_1(u,v), \end{aligned}\right\} \tag{155}$$

when $u > v$, and

$$\left.\begin{aligned} \frac{du}{d\tau} &= M\frac{u}{v}\left\{\left[1 - \left(\frac{v}{u}\right)^2\right] K\left(\frac{u}{v}\right) + \left(\frac{v}{u}\right)^2 E\left(\frac{u}{v}\right) - \alpha v\right. = P_2(u,v), \\ \frac{dv}{d\tau} &= N\left[E\left(\frac{u}{v}\right) - \beta v\right] = Q_2(u,v), \end{aligned}\right\} \tag{155a}$$

where $u < v$,

$$\left.\begin{aligned} \alpha &= 1 - \gamma\frac{\alpha_1}{\alpha_2} > 0, \quad \beta = 1 - \gamma\frac{\alpha_2}{\alpha_1} > 0, \\ M &= \frac{n_2^2 - k_1^2}{n_1^2} > 0, \quad N = \frac{k_2^2 - n_2^2}{n_1^2} > 0, \end{aligned}\right\} \tag{156}$$

since α_1 and α_2 have opposite signs and the partial frequencies lie inside the characteristic frequencies.

We shall examine a graph which shows the behavior of the integral

curves on the uv-plane, but shall limit ourselves to an examination of the first quadrant only, since a graph which shows the behavior of the integral curves of the remaining three quadrants is easily obtained by the use of a mirror image.*

On the uv-plane where $u > 0$, $v > 0$ the system (155) defines a continuous direction field since the right sides of these equations are continuous functions of u and v and the directions of the integral curves along the line $u = v$, determined by the first pair of equations of (155) coincide with the directions of the curves determined by the second pair. It is not difficult to see that the axes $v = 0$ for $u > 0$ and $u = 0$ for $v > 0$ are integral curves.

As we already know, the behavior of the integral curves determines the equilibrium states, the separatrices, and the limit cycles.

Let us look at the part of the uv-plane which satisfies the condition $u > v$.

By letting $z = v/u$ where $0 \leqslant z \leqslant 1$, in the determination of the coordinates of the equilibrium state we have the equations

$$E(z) - \alpha u = 0, \tag{157}$$

$$z\left[\left(1 - \frac{1}{z^2}\right)K(z) + \frac{1}{z^2}E(z) - \beta\frac{E(z)}{\alpha}\right] = 0. \tag{158}$$

The second of these decomposes into two:

$$z = 0, \tag{159}$$

and

$$\frac{\alpha}{\beta} = \frac{z^2 E(z)}{E(z) - (1 - z^2)\,K(z)}. \tag{160}$$

From Equations (157) and (159) it follows that there is an equilibrium state on the u-axis with coordinates $u = E(0)/\alpha$, $v = 0$.

To determine the roots of Equation (160) we construct the curve

$$y = \frac{z^2 E(z)}{E(z) - (1 - z^2)\,K(z)}.$$

---

* This follows from Equations (150) which are formulated for values of a and b which are negative and of opposite sign.

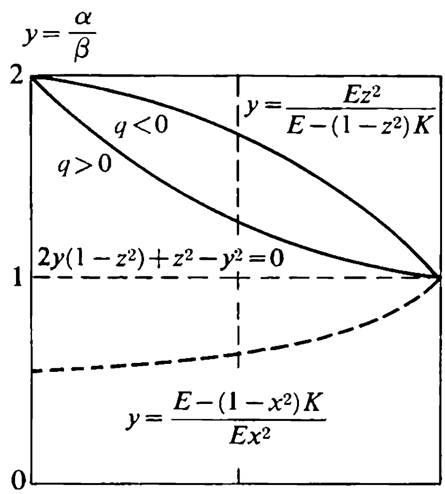


FIGURE 75

on the yz^2-plane (Figure 75).*

From Figure 75 it follows that: (a) in the range of values of z with which we are concerned Equation (160) has one positive root at $z = z_1$ where $1 \leqslant \alpha/\beta \leqslant 2$, and for all other values of α/β this equation does not have roots which are of any interest; (b) if $\alpha/\beta = 1$, the equation has a root at $z_1 = 1$; and if $\alpha/\beta = 2$, the equation has a root at $z_1 = 0$. If we know the root $z = z_1$, we can determine the coordinates of the equilibrium state by Equation (157).

Our study of the part of the plane where $v > u$ is analogous. By denoting $x = u/v$ where $0 \leqslant x \leqslant 1$, we can find the coordinates of the equilibrium states from the equations:

---

* In constructing this curve it is useful to recall the formulas

$$\frac{dK}{dz^2} = \frac{1}{2}\frac{B(z)}{1 - z^2}$$

and

$$\frac{dE}{dz^2} = -\frac{1}{2}D(z),$$

where

$$B(z) = \int_0^{\frac{\pi}{2}} \frac{\cos^2\psi\, d\psi}{\sqrt{1 - z^2\sin^2\psi}}$$

and

$$D(z) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\psi\, d\psi}{\sqrt{1 - z^2\sin^2\psi}}.$$

$$E(x) - \beta v = 0, \tag{161}$$

and

$$x\left[\left(1 - \frac{1}{x^2}\right)K(x) + \frac{1}{x^2}E(x) - \frac{\alpha E(x)}{\beta}\right] = 0. \tag{162}$$

The second of these decomposes into two more:

$$x = 0, \tag{163}$$

$$\frac{\beta}{\alpha} = \frac{x^2 E(x)}{E(x) - (1 - x^2)K(x)}. \tag{164}$$

From Equations (161) and (163) it follows that there is an equilibrium state on the v-axis with the coordinates $u = 0$, $v = E(0)/\beta$.

From an examination of the same diagram we see that Equation (164) has one and only one positive root at $x = x_1$ where $1 \leqslant \beta/\alpha \leqslant 2$, and does not have other roots for values of β/α which are of any interest; for $\beta/\alpha = 1$ we have $x_1 = 1$, and for $\beta/\alpha = 2$ we have $x_1 = 0$.

In Figure 75 the curve

$$y = \frac{E(x) - (1 - x^2)K(x)}{x^2 E(x)},$$

where $y = \alpha/\beta$, is drawn in a dotted line.

If we know the root at $x = x_1$, we can find the coordinates of the equilibrium state by using Equation (161).

Therefore, in this quadrant which includes the axes (we are excluding the point $u = 0$, $v = 0$) there can be:

a) two equilibrium states on the axes if

$$0 < \frac{\alpha}{\beta} < \frac{1}{2} \text{ and } 2 < \frac{\alpha}{\beta} < \infty;$$

b) three equilibrium states, two on the axes and one inside the axes, when

$$\frac{1}{2} < \frac{\alpha}{\beta} < 2.$$

The coordinates u_3, v_3 of this third equilibrium state satisfy the condition $u_3 > v_3$ when $1 < \alpha/\beta < 2$ and the condition $u_3 < v_3$ when $\frac{1}{2} < \alpha/\beta < 1$. And when $\alpha/\beta = 1$, $u_3 = v_3$.

Now we proceed to the investigation of the type of the equilibrium states.

As was shown in Section 3, the type of the equilibrium states of the

equations

$$\frac{du}{d\tau} = P(u,v), \quad \frac{dv}{d\tau} = Q(u,v)$$

are determined in the simplest cases, which are of interest to us, by the quantities

$$p = -[P_u'(u_0,v_0) + Q_v'(u_0,v_0)],$$

$$q = P_u'(u_0,v_0)\, Q_v'(u_0,v_0) - P_v'(u_0,v_0)\, Q_u'(u_0,v_0),$$

where u_0,v_0 are the coordinates of the equilibrium state.

For the equilibrium state on the u-axis

$$u_1 = \frac{E(0)}{\alpha} = \frac{\pi}{2\alpha}, \quad v_1 = 0,$$

$$p = M\alpha - N\left(\frac{\pi}{4u_1} - \beta\right), \quad q = -MN\alpha\left(\frac{\pi}{4u_1} - \beta\right).$$

When $4u_1/\pi < 1/\beta$, that is, when $\alpha/\beta > 2$, $q < 0$, and, consequently, this equilibrium state is a saddle point.

When $4u_1/\pi > 1/\beta$, that is, when $\alpha/\beta < 2$, we have that $q > 0$, and, likewise, $p > 0$, and the equilibrium state will be a stable node (there cannot be a focus since the u-axis is an integral curve and passes through this equilibrium state).

By analogous considerations for the equilibrium state on the v-axis we find that when $4v_2/\pi < 1/\alpha$, that is, when $\alpha/\beta < \frac{1}{2}$, it is a saddle point and, when $4v_2/\pi > 1/\alpha$, that is, when $\alpha/\beta > \frac{1}{2}$, it is a node.

For the equilibrium state which lies inside the axes in the case $u_3 > v_3$, we have

$$p = M\left(\frac{E-K}{u} + \alpha\right) + N\left[\frac{u(E-K)}{v^2} + \beta\right],$$

$$q = MN\left[\frac{(E-K)}{u}\left(\beta + \alpha\frac{u^2}{v^2}\right) + \alpha\beta\right].$$

Making use of the Equations (157) and (160) we can represent the equation for q in the form

$$q = \frac{MN}{1-z^2}\beta^2[2y(1-z^2) + z^2 - y^2]\left(y = \frac{\alpha}{\beta}\right).$$

The values of z which appear in this equation must satisfy Equation (160).

Let us construct the curve $q = 0$ on the yz^2-plane, that is,

$$2y(1 - z^2) + z^2 - y^2 = 0.$$

This curve separates those points on the plane for which $q > 0$ from those points for which $q < 0$. A drawing of this curve is shown in Figure 75. From the figure it is obvious that when $1 < \alpha/\beta < 2$, $q < 0$, and therefore the equilibrium state between the u and v-axes is a saddle point. For the case $u_3 < v_3$ one can show analogously that the equilibrium state between the axes will again be a saddle point.

Therefore the number and type of equilibrium state is uniquely determined by the ratio α/β. Table 1 summarizes our investigation.

Table 1

$\frac{\alpha}{\beta}$	the equilibrium state on the u-axis	the equilibrium state on the v-axis	the equilibrium state inside the axes
$0 < \frac{\alpha}{\beta} < \frac{1}{2}$	stable node	saddle point	none
$\frac{1}{2} < \frac{\alpha}{\beta} < 2$	stable node	stable node	saddle point
$2 < \frac{\alpha}{\beta} < \infty$	saddle point	stable node	none

So far we have discussed only points of the quadrant which lie inside the axes or on them, but we have excluded the point $u = 0, v = 0$. At this point the Equations (155) demand special attention since the values of u and v are indeterminate and, consequently do not make sense on the right-hand sides of these equations. Since the Equations (148) are satisfied by the solution $u = 0, v = 0$, or, similarly, by $a = 0, b = 0$, it is expedient to define the right-hand sides of the system (155) in such a way that the point $u = 0, v = 0$ will be an equilibrium state.*

In a neighborhood of this equilibrium state the character of the trajectories will be similar to the character of trajectories in the neighborhood of an unstable node.

Let us go on to the investigation of the behavior of the integral curves for the system (155) in the whole quadrant.

*This mathematical device simplifies the study of Equation (155). However, we note that in a neighborhood of the point $u = 0, v = 0$, there arises some doubt concerning the possibility of using Equation (155) as an approximate analysis of the system (148), because with oscillations of sufficiently small amplitudes, one can no longer consider the impulses which must satisfy the condition $I_0 n_1/c_1 \ll 1$, as small.

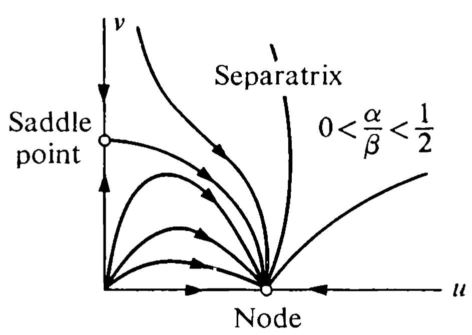


FIGURE 76

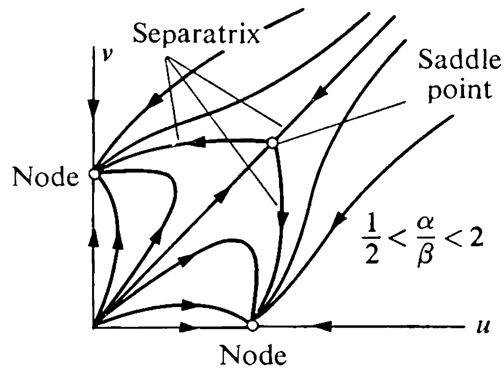


FIGURE 77

In the first place, it is not difficult to see that there will be no limit cycles in the quadrant, since inside a limit cycle there must be at least one equilibrium state and it cannot be a saddle point or an equilibrium state through which pass integral curves which are divergent to infinity.

Secondly, it is easy to see that for sufficiently large u and v and for all the values of α and β which interest us, $du/d\tau < 0$ and $dv/d\tau < 0$, that is, the motion along the integral curves for sufficiently large u and v is directed inward.

For a complete explanation of the qualitative picture of the subdivision of the quadrant into its integral curves, it remains only to explain the behavior of the separatrices.

1. When $0 < \alpha/\beta < \frac{1}{2}$, the separatrices are generated by the equilibrium state on the v-axis. It is interesting that the branch which leaves the saddle point cannot approach infinity, cannot wind around a cycle (because there are no cycles), and cannot approach the origin. Consequently, it must approach the stable node on the u-axis.

In this case the whole quadrant is a region which is "stable in the large" relative to the stable equilibrium state located on the u-axis. A qualitative picture of the uv-plane is given in Figure 76.

2. When $\frac{1}{2} < \alpha/\beta < 2$, the separatrices are generated by the equilibrium state which is inside the quadrant. It is obvious that one of the unstable branches will approach the stable equilibrium state on the u-axis, and the other will approach the stable equilibrium state on the v-axis. As regards the stable branches, one comes from infinity and the other from the origin.

The quadrant is separated into two regions of "stability in the large," which correspond to the regions of attraction of the two equilibrium states on the axes (Figure 77).

3. When $2 < \alpha/\beta < \infty$, the separatrix issuing from the saddle point

on the u-axis is connected to the stable node on the v-axis. The whole quadrant is a region of "stability in the large" for this equilibrium state (Figure 78).

By making use of the foregoing discussion of the plane with the variables u and v, we can assert that the motions of most interest to us are the periodic motions which are approximated by the normal modes with frequencies of k_1 and k_2. On the uv-plane these motions correspond to the equilibrium states located on the axes. The saddle point between the axes corresponds to unstable biharmonic motion which is of physical interest only in its connection with the separatrix surfaces which divide the state space into regions of "stability in the large," which, in turn, are associated with motions that are approximately harmonic.

In order to clarify the visual picture of the phenomena which can take place in this dynamic system, we shall vary the partial frequency of the clock frame by gradually changing the distance between the points of suspension, if the clock is suspended by two threads; the other quantities which characterize the systems shall remain the same.

By way of a parameter, whose variation expresses the variation of the distance between the points of suspension, we take the square of the quotient of the partial frequency of the clock frame and the partial frequency of the balance wheel, $\zeta = n_2^2/n_1^2$.

We can get a visual picture of the phenomena that take place in the system as ζ is gradually varied in both directions if we construct graphs of the quantities as a function of the variable ζ: the curves of the frequencies and amplitudes of the balance wheel and of the clock frame. On these graphs we can show portions which correspond to stable and unstable motion and also to "jumps," which determine the behavior of the corresponding regions of "stability in the large."

We shall assume the following quantities are fixed:

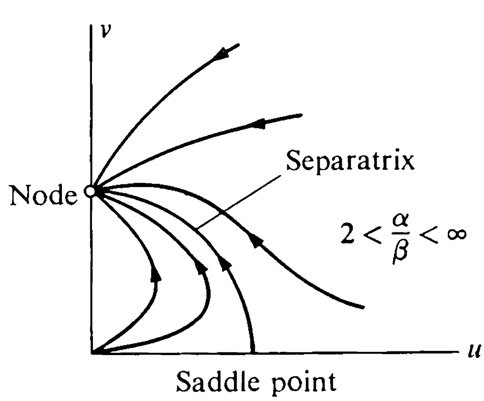


FIGURE 78

$$\gamma_0 = \gamma\zeta \left(\gamma_0 = \frac{b_2 J_1}{b_1 (J_1 + J_2 + mh^2)}\right), \sigma = 1 - |A_1 A_2.$$

If in the formulas

$$u_1 = \frac{E(0)}{\alpha} = \frac{\pi}{2\alpha}, \quad v_2 = \frac{E(0)}{\beta} = \frac{\pi}{2\beta}$$

we let $u_1 = (\pi/2)\, Z_1$ and $v_2 = (\pi/2)\, Z_2$, then we get the values of the amplitudes of the balance wheel for periodic motions, $Z_1 = 1/\alpha$ and $Z_2 = 1/\beta$.

According to (156) we have $\alpha = 1 - \gamma(\alpha_1/\alpha_2)$, and $\beta = 1 - \gamma(\alpha_2/\alpha_1)$. The coefficients α_1 and α_2 are the roots of the equation

$$A_1 n_1^2 \alpha^2 + (n_2^2 - n_1^2)\,\alpha - n_1^2 A_2 = 0$$

or,

$$A_1 \alpha^2 + (\zeta - 1)\,\alpha - A_2 = 0.$$

By finding these roots and substituting in α and β, we get expressions for the amplitudes of the balance wheel,

$$Z_1 = \frac{1}{1 - \gamma_0 M(\zeta)}, \quad Z_2 = \frac{1}{1 - \gamma_0 N(\zeta)},$$

where

$$M(\zeta) = \frac{\zeta - 1 - \sqrt{(1 + \zeta)^2 - 4\sigma\zeta}}{\zeta\left(\zeta - 1 + \sqrt{(1 + \zeta)^2 - 4\sigma\zeta}\right)},$$

$$N(\zeta) = \frac{\zeta - 1 + \sqrt{(1 + \zeta)^2 - 4\sigma\zeta}}{\zeta\left(\zeta - 1 - \sqrt{(1 + \zeta)^2 - 4\sigma\zeta}\right)}.$$

The periodic motions with the frequencies k_1 and k_2 are stable when

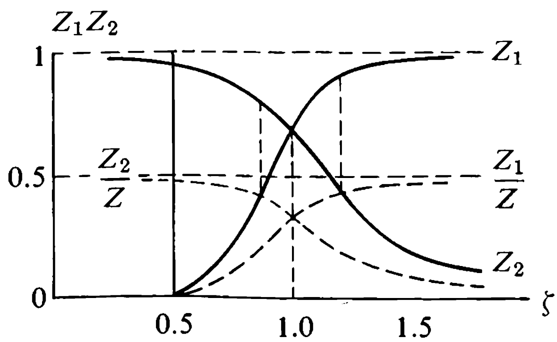


FIGURE 79

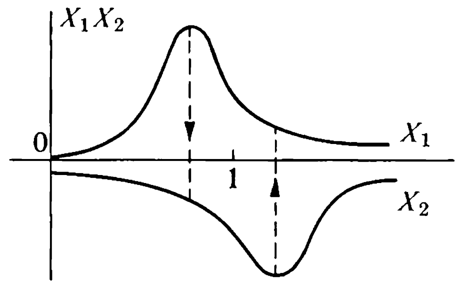


FIGURE 80

$Z_1/Z_2 > ½$ (that is, when $\alpha/\beta < 2$) and when $Z_2/Z_1 > ½$ (that is, when $\alpha/\beta > ½$), respectively.

The formulas for the amplitude of the clock frame are

$$X_1 = \frac{\alpha_1}{1 - \gamma_0 M(\zeta)}, \quad X_2 = \frac{\alpha_2}{1 - \gamma_0 N(\zeta)}.$$

The curve for the frequencies is given by the formula

$$\sigma\Omega^4 - (1 + \zeta)\,\Omega^2 + \zeta = 0 \quad \left(\Omega = \frac{k}{n_1}\right).$$

The curves for Z_1, Z_2, X_1, X_2, Ω_1^2, Ω_2^2, when $\sigma = 0.99$ and $\gamma_0 = 0.5$, are shown in Figures 79, 80, and 81. The stable portions of these curves are indicated by broad lines. The curves for other values of σ and γ_0 are analogous. These drawings visually demonstrate the phenomenon of "entrainment" which takes place in the clock when the rigidity of its suspension is varied. This phenomenon consists in the fact that the jump from one normal frequency to another does not take place when $\zeta = 1$, but depends on the "history" of the variation in ζ.

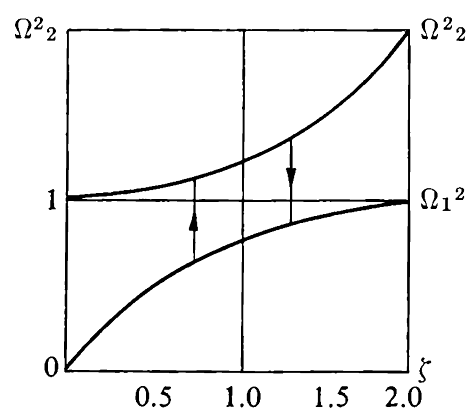


FIGURE 81

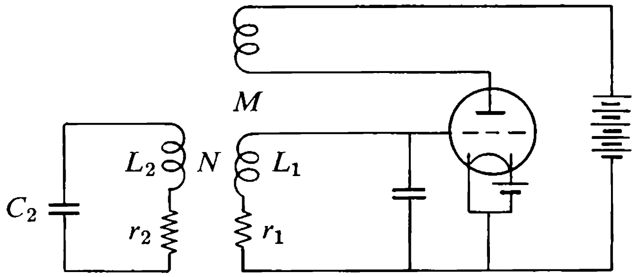


FIGURE 82

In some interval of variation in ζ it is possible to have oscillations involving both normal frequencies (it is more accurate to say approximately normal frequencies).

In conclusion, we note that experimental studies of this phenomenon [12], [56] qualitatively verify the theory developed in this section. It would be impossible to demand a quantitative agreement of the theory with experiment, because of the assumptions on which theory is based and, most of all, because the forces of friction and the operation of the suspension mechanism greatly differ from the way in which a pocket watch works.

We note that in R. Pol's book, *An Introduction to Mechanics and Acoustics,* ONTI, 1933, page 224, he describes an experiment which must be explained, essentially, with the help of the theory developed here.

By way of a second example we shall study self-excited oscillations in a complex vacuum-tube circuit [34].

Let there be given two inductively coupled circuits, one of which is connected to the grid of a triode.

For the two circuits, which are sketched in Figure 82, using Kirchoff's laws and neglecting the grid current, one can write

$$\left.\begin{aligned} L_1\frac{di_1}{dt} + r_1 i_1 + \frac{1}{c_1}\int_0^t i_1\,dt &= M\frac{di_a}{dt} + N\frac{di_2}{dt}, \\ L_2\frac{di_2}{dt} + r_2 i_2 + \frac{1}{c_2}\int_0^t i_2\,dt &= N\frac{di_1}{dt}. \end{aligned}\right\} \tag{165}$$

If we neglect the plate resistance and assume that the plate current-grid voltage characteristic is a cubic, then

$$i_a = Sv_1\left(1 - \frac{v_1^2}{3K^2}\right),$$

where

$$v_1 = \frac{1}{c_1}\int_0^t i_1\, dt;$$

S is the steepness of this characteristic, and K is the saturation voltage.

We introduce the notation

$$v_2 = \frac{1}{c_2}\int_0^t i_2\, dt, \quad n_1^2 = \frac{1}{L_1c_1}, \quad n_2^2 = \frac{1}{L_2c_2}, \quad \lambda_1 = Nc_2n_1^2, \quad \lambda_2 = Nc_1n_2^2\,,$$

$$v_1 = xK\sqrt{\frac{MS - c_1r_1}{MS}}, \quad v_2 = yK\sqrt{\frac{MS - c_1r_1}{MS}}$$

and rewrite Equation (165) in the form

$$\ddot{x} - \lambda_1\ddot{y} + n_1^2x = \mu f\,(x,\dot{x},y,\dot{y}),$$

$$\ddot{y} - \lambda_2\ddot{x} + n_2^2y = \mu g\,(x,\dot{x},y,\dot{y}),$$

where

$$\mu = n_1\,(MS - c_1r_1),\ f = n_1\,(1 - x^2)\,\dot{x},$$

$$g = -\delta\frac{n_2^2}{n_1^2}\dot{y}, \quad \delta = \frac{c_2r_2}{MS - c_1r_1}.$$

In this case, Equation (144) becomes

$$\left.\begin{aligned} \frac{du}{d\tau} &= Au\,(\alpha - u - 2v), \\ \frac{dv}{d\tau} &= Bv\,(\beta - 2u - v), \end{aligned}\right\} \tag{166}$$

where

$$u = a^2, \quad v = b^2, \quad A = \frac{1}{2}\frac{n_2^2 - k_1^2}{n_1^2} > 0, \quad B = \frac{1}{2}\frac{k_2^2 - n_2^2}{n_1^2} > 0,$$

$$\alpha = 4 - 4\delta\frac{n_2^2}{n_1^2}\frac{n_1^2 - k_1^2}{n_2^2 - k_1^2}, \quad \beta = 4 - 4\delta\frac{n_2^2}{n_1^2}\frac{k_2^2 - n_1^2}{k_2^2 - n_2^2},$$

$$\tau = \frac{\mu n_1^3 t}{2\,(k_2^2 - k_1^2)}.$$

From (145) it follows that β_1 is constant and β_2 is constant.

For the solution of the problem of the possible forms of motions we investigate the possible ways of subdividing the uv-state plane into trajectories, which, in turn, reduces to the task of finding and studying the singular points, limit cycles, and separatrices of Equation (166).

Only the first quadrant of the uv-plane will be of interest to us since $u = a^2 > 0$ and $v = b^2 > 0$.

The singular points and the roots of their corresponding characteristic equations are listed below.

The point P_1: $u_1 = 0$, $v_1 = 0$ and $s_1 = A\alpha$, $s_2 = B\beta$. Because of (130) this point corresponds to an equilibrium state in the original system.

The point P_2: $u_2 = \alpha$, $v_2 = 0$, $s_1 = -A\alpha$, $s_2 = B(\beta - 2\alpha)$.

In the original system this point corresponds to harmonic motion with a frequency of k_1.

The point P_3: $u_3 = 0$, $v_3 = \beta$, $s_1 = A(\alpha - 2\beta)$, $s_2 = -B\beta$.

In the original system this point corresponds to harmonic motion with a frequency of k_2.

The point P_4: $u_4 = 2\beta - \alpha/3$, $v_4 = 2\alpha - \beta/3$.

When $u_4 > 0$, $v_4 > 0$, the roots of the characteristic equation have opposite signs so that this point is always unstable (a saddle point). In the original system it corresponds to biharmonic motion and is unstable.

Since $u = 0$, $v = 0$ are solutions and inside the quadrant $u \geqslant 0$, $v \geqslant 0$ there cannot be any singular points except a saddle point, it follows that there is no limit cycle. For large u and v, we have

$$\frac{du}{d\tau} < 0, \quad \frac{dv}{d\tau} < 0,$$

that is, the singular point at infinity is unstable.

The character of the subdivision of the uv-plane into trajectories is determined by the values of the quantities α and β, which, in turn, depend on the parameters of the system. Without studying how α and β depend

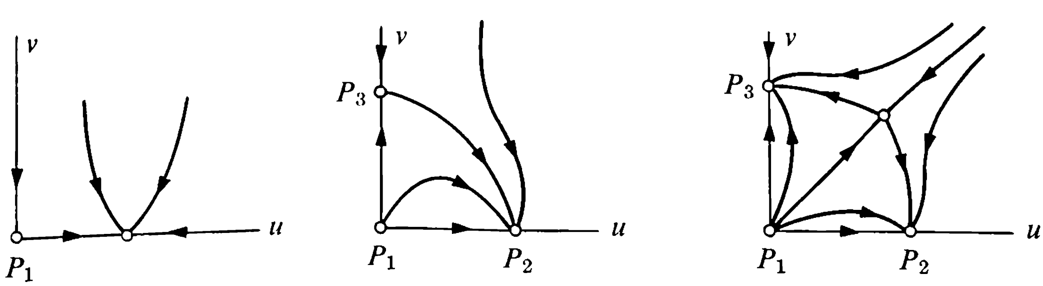


FIGURE 83

on the parameters of the system in this example, we point out that the possible forms of the state plane are shown in Figure 83.

10. Systems With Gyroscopic Forces

In this section we shall study dynamic systems with gyroscopic forces which are approximated by a linear conservative system.

The equations of motion of such a system with two degrees of freedom have the form

$$\left.\begin{aligned} \ddot{x} - \lambda_1 \dot{y} + (\pm n_1^2)\, x &= \mu f(x, \dot{x}, y, \dot{y}), \\ \ddot{y} + \lambda_2 \dot{x} + (\pm n_2^2)\, y &= \mu g\,(x, \dot{x}, y, \dot{y}), \end{aligned}\right\} \tag{167}$$

where λ_1, λ_2, n_1^2, n_2^2 are constants, μ is a small parameter which characterizes the closeness of the system to a linear conservative one, and f and g are nonlinear functions.

The terms $\lambda_1 y$ and $\lambda_2 x$ are called gyroscopic. This designation can be explained by the presence of this type of term in the equations for the motion of a gyroscope [30].

When $\mu = 0$, the system (167) has the solution

$$\left.\begin{aligned} x &= a \sin(k_1 t + \beta_1) + b \sin(k_2 t + \beta_2), \\ y &= \alpha_1 a \cos(k_1 t + \beta_1) + \alpha_2 b \cos(k_2 t + \beta_2), \end{aligned}\right\} \tag{168}$$

where k_1 and k_2 are characteristic (normal) frequencies which are roots of the equations

$$k^4 + [(\mp n_1^2) + (\mp n_2^2) - \lambda_1 \lambda_2]\, k^2 + n_1^2 n_2^2 = 0. \tag{169}$$

The coefficients α_1, α_2 are determined by the formulas

$$\left.\begin{aligned} \alpha_1 &= \frac{(\mp n_1^2) + k_1^2}{\lambda_1 k_1} = \frac{\lambda_2 k_1}{(\mp n_2^2) + k_1^2}, \\ \alpha_2 &= \frac{(\mp n_1^2) + k_2^2}{\lambda_1 k_2} = \frac{\lambda_2 k_2}{(\mp n_2^2) + k_2^2}. \end{aligned}\right\} \tag{170}$$

We are looking for a solution of the system (167) when $\mu \neq 0$ of the form (168), but we are assuming that α and β are already slowly changing functions with respect to time.

After having made calculations analogous to those carried out in the

previous section, one can prove that the functions a and b must satisfy the following system of equations:

$$\left.\frac{da}{d\tau} = -\frac{1}{n_1^3}\left(\lambda_1 G_1 - \frac{\lambda_2}{\alpha_1}F_1\right); \quad \frac{db}{d\tau} = \frac{1}{n_1^3}\left(\lambda_1 G_2 - \frac{\lambda_2}{\alpha_2}F_2\right).\right\} \tag{171}$$

Here,

$$\left.\begin{aligned} G_1 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\xi \, d\xi \, d\eta; \quad F_1 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\xi \, d\xi \, d\eta, \\ G_2 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\eta \, d\xi \, d\eta; \quad F_2 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\eta \, d\xi \, d\eta, \end{aligned}\right\} \tag{172}$$

where

$$\begin{aligned} f^* &= f(a\sin\xi + b\sin\eta, \quad ak_1\cos\xi + bk_2\cos\eta, \\ &\qquad \alpha_1 a\cos\xi + \alpha_2 b\cos\eta, \quad -\alpha_1 k_1 a\sin\xi - \alpha_2 k_2 b\sin\eta); \\ g^* &= g(a\sin\xi + b\sin\eta, \quad ak_1\cos\xi + bk_2\cos\eta, \\ &\qquad \alpha_1 a\cos\xi + \alpha_2 b\cos\eta, \quad -\alpha_1 k_1 a\sin\xi - \alpha_2 k_2 b\sin\eta); \end{aligned}$$

$$\xi = k_1 t + \beta_1, \quad \eta = k_2 t + \beta_2;$$

$$\tau = \frac{\mu n_1^3 t}{2(k_1^2 - k_2^2)} \quad (k_1^2 > k_2^2).$$

As an example we shall study a monorail with a gyroscopic stabilizer. For small oscillations the differential equations of such a system are

$$\left.\begin{aligned} A_0\ddot{x} - C\omega\dot{y} - pbx &= -\gamma''\dot{x} + M_x(x,\dot{x},y,\dot{y}), \\ J_0\ddot{y} + C\omega\dot{x} - P_1c_1y &= -\gamma'\dot{y}, \end{aligned}\right\} \tag{173}$$

where x is the angle of rotation of the frame of the gyroscope, y is the angle of displacement of the monorail from a vertical, ω is the angular velocity of the gyroscope itself, and M_x is the servotorque which stabilizes the monorail.*

Servotorques M_x can have several forms:

1) $M_x = \pm\Phi_0$, where Φ_0 is constant. It has a plus sign when $y > 0$, and

* A derivation of Equations (173) and an explanation of the meanings of the quantities in it are given in a book by L. G. Loytsyanskiy and A. I. Lur'e [30].

a minus sign when $y < 0$. Action of this type is approximated by Shilovskiy's model.†

2) $M_x = I_0 \delta(x)\dot{x}$, where I_0 is the absolute value of the moment of the impact and $\delta(x)$ is Dirac's delta function. This moment is analogous to the impact which effects stabilization in one of Sherl's patented devices.

3) $M_x = (\alpha' - \beta'\dot{x}^2)\dot{x}$, where $\alpha' > 0$, $\beta' > 0$. Such a torque can be realized by using an asynchronous motor to rotate the frame of the gyroscope with the help of a bevel gear.

4) $M_x = -(\alpha' - \beta' y^2) y$, where $\alpha' > 0$, $\beta' > 0$. Such a torque can be realized by using, for example, a pendulum type apparatus which will measure the angle of displacement of the monorail and which is connected to a relay.

We shall limit ourselves to an examination of only one case, namely, when*

$$M_x = (\alpha' - \beta'\dot{x}^2)\dot{x}.$$

We rewrite Equation (173) in the form

$$\left.\begin{aligned} \ddot{x} - \lambda_1 \dot{y} - n_1^2 x &= -\frac{\gamma''}{A_0}\dot{x} + \frac{1}{A_0}(\alpha' - \beta'\dot{x}^2)\,x, \\ \ddot{y} + \lambda_2 \dot{x} - n_2^2 y &= -\frac{\gamma'}{J_0}\dot{y}. \end{aligned}\right\} \qquad (174)$$

† One can study the discontinuous function $M_x = \Phi_0$ as a limit of a sequence of continuous functions

$$M_x^{(i)} = \frac{2\Phi_0}{\pi}\arctan b_i y$$

where $b_i \to \infty$, and the improper function $M_x = I_0 \dot{x}\delta(x)$ as the limit of a sequence of continuous functions

$$M_x^{(i)} = \frac{I_0}{\pi}\frac{a_i}{a_i + x^2}\dot{x}$$

where $a_i \to 0$. Such limit functions are often used in the theory of oscillations. However, one should keep in mind two facts. First, real servotorques are better idealized by functions in these sequences, rather than their limits. Second, results attained by the method of slowly varying parameters (van der Pol's method) for these kinds of discontinuous characteristics should be considered as results which are valid for continuous systems which closely approximate them. The introduction of discontinuous characteristics is, essentially, a device for simplifying the analysis, and, in particular, for computing all the necessary integrals. They are not worthwhile if they introduce special complications of their own.

* In [13] the other cases are studied.

Here,

$$\lambda_1 = \frac{C\omega}{A_0}, \quad \lambda_2 = \frac{C\omega}{J_0}, \quad n_1^2 = \frac{pb}{A_0}, \quad n_2^2 = \frac{P_1 c_1}{J_0}.$$

We assume that the dimensionless quantities satisfy

$$\frac{\alpha' - \gamma''}{A_0 n_1} \ll 1, \quad \frac{\gamma'}{J_0 n_1} \ll 1, \quad \frac{\beta' n_1}{A_0} \ll 1.$$

Then, the equations for the motion of the system take the form

$$\left.\begin{aligned} \ddot{x} - \lambda_1 \dot{y} - n_1^2 x &= \mu n_1 (\dot{x} - \beta \dot{x}^3), \\ \ddot{y} + \lambda_2 \dot{x} - n_2^2 y &= \mu n_1 (-\lambda \dot{y}), \end{aligned}\right\} \tag{175}$$

where

$$\mu = \frac{\alpha' - \gamma''}{A_0 n_1}, \quad \beta = \frac{\beta'}{\alpha' - \gamma''}, \quad \lambda = \frac{A_0 \gamma'}{J_0 (\alpha' - \gamma'')}.$$

The Equations (170) in this case become simple enough so that the integrals (172) can be computed rather easily.

If we introduce the notation

$$A = \frac{3\beta\lambda_2 k_1}{2n_1^2 \alpha_1}, \quad B = \frac{3\beta\lambda_2 k_2}{2n_1^2 \alpha_2}, \quad u = a^2 k_1^2, \quad v = b^2 k_2^2,$$

$$\alpha_0 = \frac{4}{3\beta}\left(1 - \lambda \frac{\lambda_1}{\lambda_2} \alpha_1^2\right), \quad \beta_0 = -\frac{4}{3\beta}\left(1 - \lambda \frac{\lambda_1}{\lambda_2} \alpha_2^2\right),$$

where k_1^2 and k_2^2 are the roots of the frequency equation*

$$k^4 + (n_1^2 + n_2^2 - \lambda_1 \lambda_2) k^2 + n_1^2 n_2^2 = 0,$$

and, where α_1 and α_2 are given by the equations

$$\alpha_1 = \frac{n_1^2 + k_1^2}{\lambda_1 k_1} = \frac{\lambda_2 k_1}{n_2^2 + k_1^2},$$

$$\alpha_2 = \frac{n_1^2 + k_2^2}{\lambda_1 k_2} = \frac{\lambda_2 k_2}{n_2^2 + k_2^2},$$

then Equations (170) take the form

---

* In order that the linear system $\ddot{x} - \lambda_1 \dot{y} - n_1^2 x = 0$, $\ddot{y} + \lambda_2 \dot{x} - n_2^2 y = 0$, have a solution of the form (168), the roots of the characteristic equation must be real and positive. This happens when $n_1^2 + n_2^2 - \lambda_1 \lambda_2 < 0$ and $(n_1^2 + n_2^2 - \lambda_1 \lambda_2)^2 - 4 n_1^2 n_2^2 > 0$.

We shall study only those values of λ_1, λ_2, n_1^2, and n_2^2, which satisfy these inequalities.

$$\left.\begin{aligned}\frac{du}{d\tau} &= Au[\alpha_0 - u - 2v] = P(u,v),\\ \frac{dv}{d\tau} &= Bv[\beta_0 + 2u + v] = Q(u,v).\end{aligned}\right\} \tag{176}$$

The singular points of the system (174) are given by the equations

$$u(\alpha_0 - u - 2v) = 0,$$
$$v(\beta_0 + 2u + v) = 0.$$

The coordinates of the singular points are as follows:

1) point P_1 at $u_1 = 0$, $v_1 = 0$. For the system (174) according to (168) this point corresponds to a state of equilibrium.

2) point P_2 at $u_2 = \alpha_0$, $v_2 = 0$. For the system (174) this point corresponds to periodic motion with a frequency of k_1.

3) point P_3 at $u_3 = 0$, $v_3 = -\beta_0$. For the system (174) this point corresponds to periodic motion with a frequency of k_2.

4) point P_4 at

$$u_4 = -\frac{2\beta_0 + \alpha_0}{3}, \quad v_4 = \frac{2\alpha_0 + \beta_0}{3}.$$

For the system (174) this point corresponds to biharmonic motion with frequencies k_1 and k_2.

We shall examine only the first quadrant of the uv-plane since $u = a^2k_1^2$, $v^2 = b^2k_2^2$.

We shall find the conditions for the stability of singular points.

The characteristic equation, the roots of which determine the type of the singular points, has the form (see Section 3)

$$s^2 - (a + d)s + ad - bc = 0,$$

where

$$a = P'_u(u_0,v_0), \quad c = Q'_u(u_0,v_0),$$
$$b = P'_v(u_0,v_0), \quad d = Q'_v(u_0,v_0)$$

(u_0,v_0 are the coordinates of the singular point).

The roots of this equation are:

for the point P_1, $s_1 = A\alpha_0$, $s_2 = B\beta_0$;

for the point P_2, $s_1 = B(2\alpha_0 + \beta_0)$, $s_2 = -A\alpha_0$;

for the point P_3, $s_1 = -B\beta_0$, $s_2 = A(\alpha_0 + 2\beta_0)$.

for the point P_4 the characteristic equation has the form

$$s^2 - (Bv_4 - Au_4)s + 3ABu_4v_4 = 0,$$

and, hence,

$$s_1 s_2 = 3ABu_4 v_4; \; s_1 + s_2 = Bv_4 - Au_4.$$

If we let

$$z_1 = \frac{3\beta}{4}\alpha_0, \quad z_2 = -\frac{3\beta}{4}\beta_0,$$

then the results of this study of the number and type of the singular points are summarized in Table 2.

We have not touched on proofs of the facts that Equations (176) do not have limit cycles and that when the stability changes the point P_4 is a center.

Since u and v are positive, for sufficiently large A and B we have

$$\frac{du}{d\tau} < 0, \quad \frac{dv}{d\tau} > 0.$$

Table 2

Case		a singular point at the coordinate origin	a singular point on the u-axis	a singular point on the v-axis	a singular point between the axes
$z_1 <$	$0, z_2 > 0$	a stable node	none	a saddle point	none
$z_1 > 0$, $z_2 > 0$	$2z_1 - z_2 > 0$, $z_1 - 2z_2 > 0$	a saddle point	a saddle point	an unstable node	none
	$2z_1 - z_2 < 0$, $z_1 - 2z_2 < 0$	a saddle point	an unstable node	a saddle point	none
	$2z_1 - z_2 > 0$, $z_1 - 2z_2 < 0$	a saddle point	a saddle point	a saddle point	a stable or unstable node (focus)*
$z_1 <$	$0, z_2 < 0$	a saddle point	none	none	none
$z_1 >$	$0, z_2 < 0$	an unstable node	a saddle point	none	none

* For $z_1 > (2A + B/2B + A)\, z_2$ there is an unstable singular point.
For $z_1 < (2A + B/2B + A)\, z_2$ there is a stable singular point.

From Table 2 we see that the motions which are of interest are the stable biharmonic motion and the stable harmonic motion with the frequency k_1, where the harmonic motion is stable for

$$z_1 < \tfrac{1}{2} z_2 \quad (z_1 > 0,\ z_2 > 0),$$

and the biharmonic motion is stable for

$$z_1 < \frac{2A + B}{2B + A} z_2.$$

The equilibrium state is stable for $z_1 < 0$, $z_2 > 0$.

We shall now investigate the dependence of the stability of the motions on the parameters of the system. To this end, we begin with the following mechanical picture: we vary the moment of inertia of the monorail while all the other quantities which characterize the system remain constant.

As a parameter whose variation will serve to express variation on the moment of inertia, we take the quotient $\zeta = n_2^2/n_1^2$. Then z_1 and z_2 will be functions of ζ and will have the form

$$z_1 = 1 - \lambda_0 M(\zeta), \quad z_2 = 1 - \lambda_0 N(\zeta),$$

$$\lambda_0 = \frac{\gamma' pb}{(\alpha' - \gamma'')\, P_1 c_1}, \quad \sigma_0 = \frac{c^2 \omega^2}{A_0 P_1 c_1},$$

where

$$M(\zeta) = \zeta \frac{(\sigma_0 - 1)\zeta + 1 + \sqrt{[(\sigma_0 - 1)\zeta - 1]^2 - 4\zeta}}{(\sigma_0 + 1)\zeta - 1 + \sqrt{[(\sigma_0 - 1)\zeta - 1]^2 - 4\zeta}},$$

$$N(\zeta) = \zeta \frac{(\sigma_0 - 1)\zeta + 1 - \sqrt{[(\sigma_0 - 1)\zeta - 1]^2 - 4\zeta}}{(\sigma_0 + 1)\zeta - 1 - \sqrt{[(\sigma_0 - 1)\zeta - 1]^2 - 4\zeta}}.$$

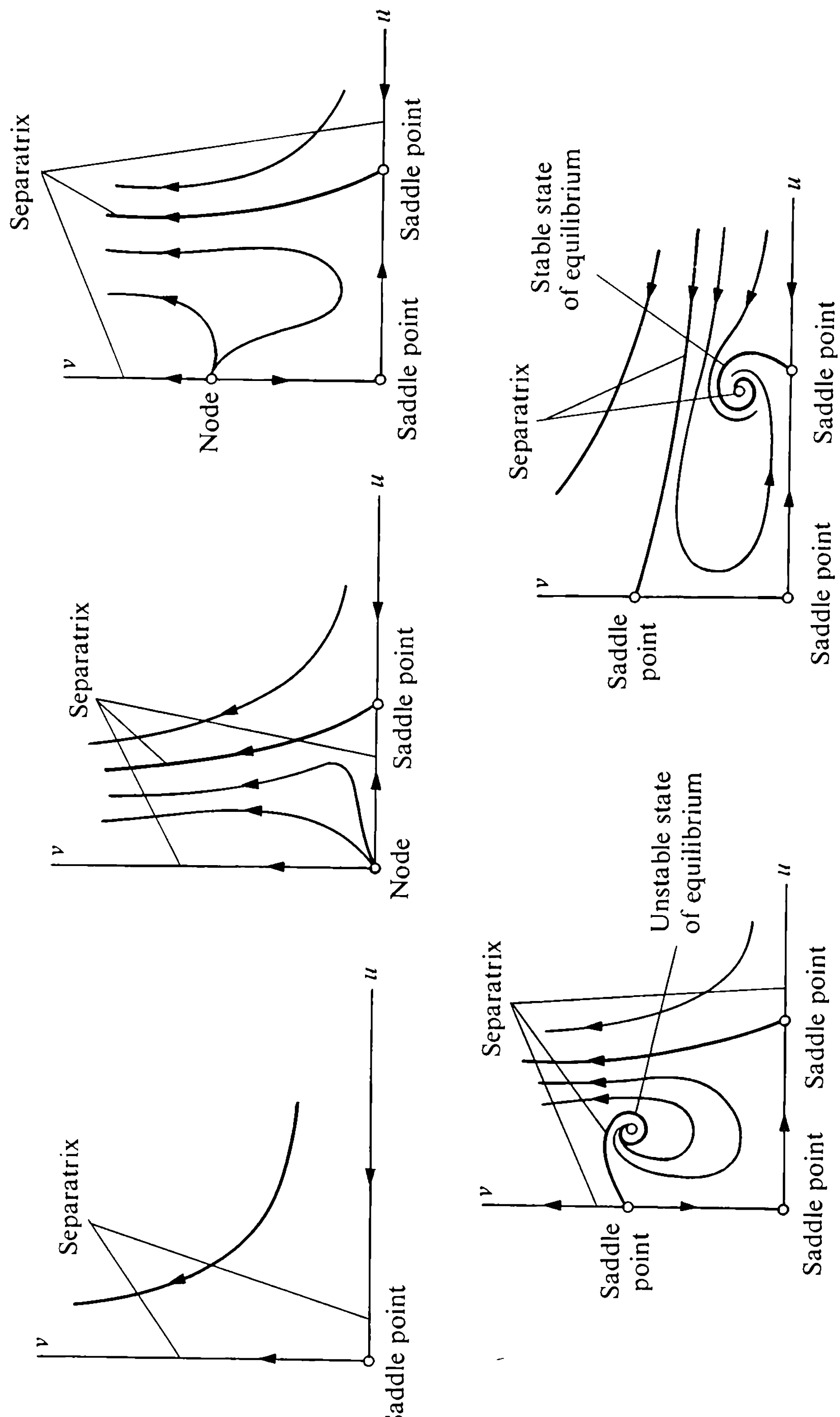
We shall examine only those values of ζ for which the Hamiltonian system is stabilized by the gyroscope. In this instance, this requirement is fulfilled when

$$\zeta > \zeta_0 = \frac{\sigma_0 + 1 + 2\sqrt{\sigma_0}}{(\sigma_0 - 1)^2}.$$

An investigation of the behavior of the functions z_1 and z_2 in their dependence on ζ gives the following results.

For $\lambda_0 < 1$ when $\zeta > 1$ the system can have stable biharmonic motion, stable harmonic motion, and a stable equilibrium state; when $\zeta > 1$, of the stable stationary motions, only a stable biharmonic motion is possible.

For $\lambda_0 > 1$ when $\zeta < 1$ there can be only a stable biharmonic motion; when $\zeta > 1$, the system does not have stable motions.


Separatrix
Saddle point
Node
Saddle point
Saddle point
v
u
Separatrix
Saddle point
Node
v
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Separatrix
Saddle point
v
u
Separatrix
Stable state
of equilibrium
Saddle
point
Saddle point
Saddle point
v
u
Separatrix
Unstable state
of equilibrium
Saddle
point
Saddle point
Saddle point
v
u

FIGURE 84

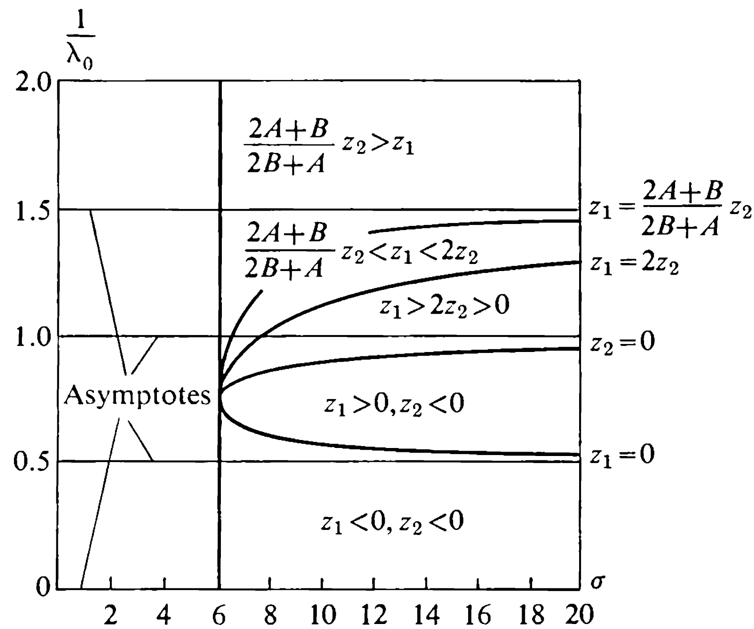

FIGURE 85

In Figure 85 the dependence of the motions of the system on the parameters σ_0, λ_0 is shown (a bifurcation diagram), and it is constructed for the case $\zeta = 0.5$. A qualitative picture on the uv-plane, which corresponds to this case is shown in Figure 84.

The bifurcation sketch in Figure 86 is constructed for the case $\zeta = 2$. Qualitative pictures of the uv-plane are given in Figure 87.

Figure 88 shows the curves z_1, z_2, $(2A + B/2B + A)\ z_2$, and is con-

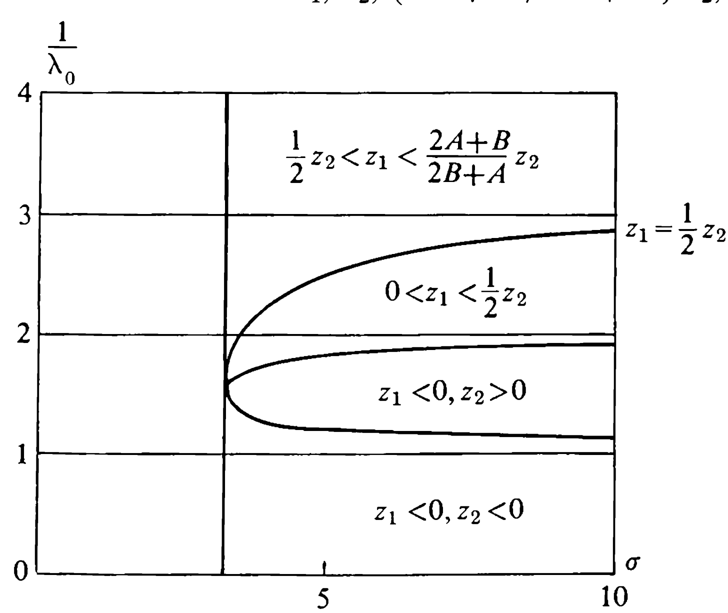

FIGURE 86

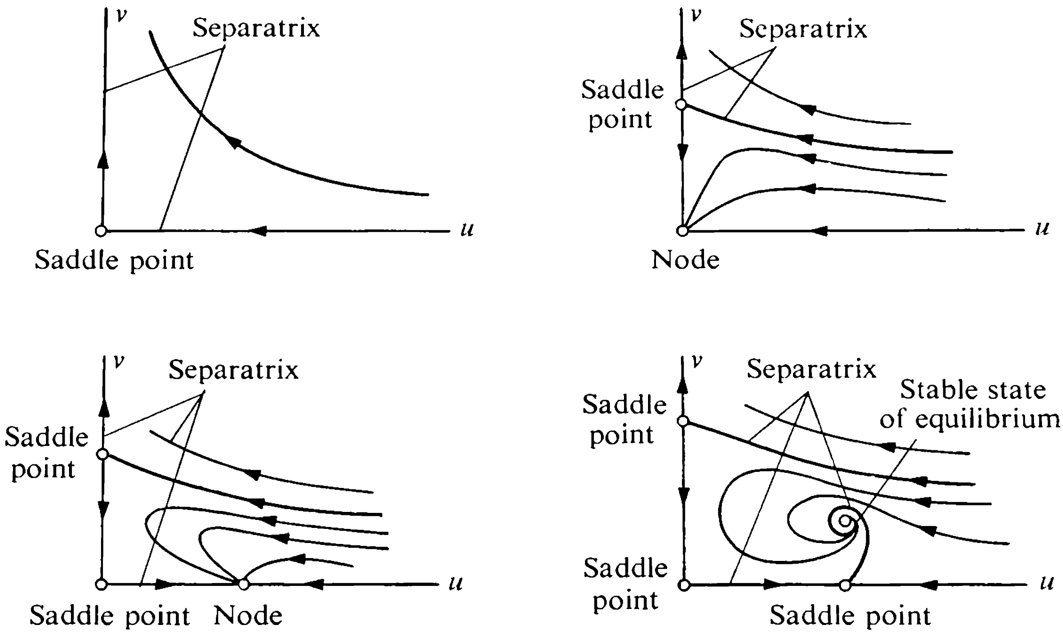

FIGURE 87

structed for fixed σ_0 and λ_0. From the drawing it is obvious that when

$$\frac{\sigma_0 + 1 + 2\sqrt{\sigma_0}}{(\sigma_0 - 1)^2} < \zeta < \zeta_1$$

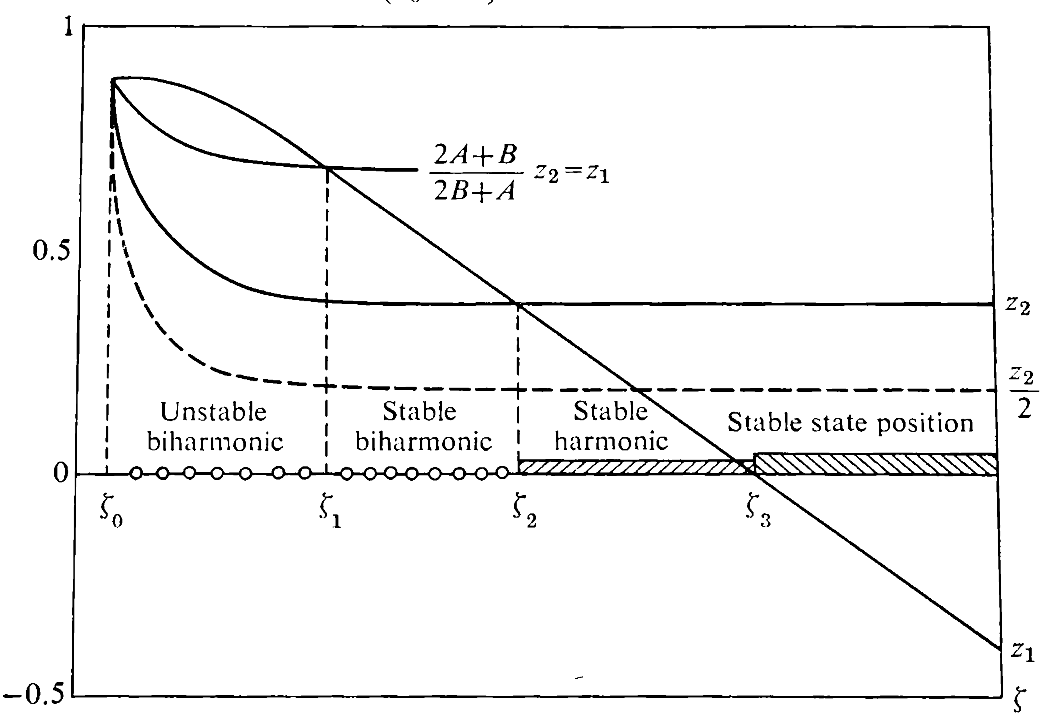

FIGURE 88

the biharmonic motion is unstable; when $\zeta_1 < \zeta < \zeta_2$, the biharmonic motion with frequencies k_1 and k_2 is stable; when $\zeta_2 < \zeta < \zeta_3$, harmonic motion with the frequency k_1 is stable; and when $\zeta > \zeta_3$, the position at rest is a stable state.

III

Degenerate Systems

11. A Study of Degenerate Systems With the Help of the "Jump" Hypothesis [2], [14]

In the study of nonlinear systems with one degree of freedom one must investigate differential equations of the second order. However, in some cases the conditions of the problem make it possible to neglect some parameter of the system. Then the differential equation of the motion becomes an equation of the first order and defines the motion of a so-called "degenerate" system.

For example, consider differential equation

$$m\ddot{x} + \varphi(\dot{x}) + cx = 0,$$

which describes the motion of a point mass m under the influence of a restoring force and in the presence of a resistance which is a function of the velocity. If the mass is negligible, then we get a differential equation of the form

$$\varphi(\dot{x}) + cx = 0.$$

By solving this equation for $\dot{x}$, we have

$$\frac{dx}{dt} = f(x).$$

If one neglects the restoring force then the differential equation has the form

$$m\ddot{x} + \varphi(\dot{x}) = 0 \quad \text{or} \quad \frac{dy}{dt} = \psi(y),$$

where $y = \dot{x}$, and

$$\psi(y) = -\frac{1}{m}\varphi(y).$$

The objective of this section is to study the motion of such degenerate systems.

Let the motion of the degenerate system be described by the equation*

$$\frac{dx}{dt} = f(x), \tag{177}$$

where $f(x)$ is an analytical function.

We shall clarify what the possible motions are in this case.

Let $f(x) = 0$ have no real roots. This means that the system described by (177) does not have any states of equilibrium. Consequently, dx/dt will always have the same sign and $x(t)$ will either be monotonically increasing or monotonically decreasing.

Now, let $f(x) = 0$ have real roots $x_1, x_2, \ldots, x_\kappa$. These roots correspond to state of equilibrium of the system.

We shall study the tx-plane. On this plane the solutions $x = x_1$, $x = x_2$, $\ldots$, $x = x_\kappa$ are lines parallel to the t-axis (Figure 89). By the theorem for the existence and uniqueness of the solution to (177) the integral curves cannot intersect one another so that all other solutions of this equation lie inside the strips defined by the solutions $x = x_1$, $x = x_2, \ldots, x = x_\kappa$ (Figure 89). Since inside any one of the strips the sign of $f(x)$ does not change, these solutions will be monotonic. This means that if $f(x)$ is an analytic function, then Equation (177) cannot have periodic solutions.

The state plane for Equation (177) degenerates into a state line. Let us look at the representation of the motion on the state line.

We note that according to the uniqueness theorem concerning the solution of Equation (177), the initial condition $x = x_0$ when $t = t_0$

*A proof of the existence and uniqueness of the solution to this equation can be found, for example, in reference [2].

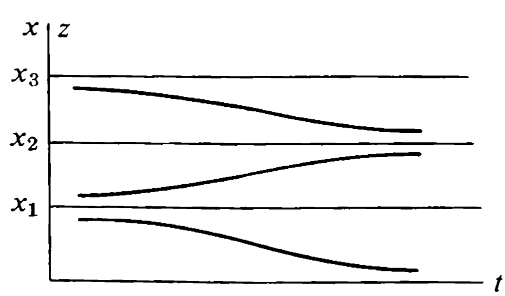

FIGURE 89

uniquely determines the further motion of the representative point. The character of the motion of the representative point will not depend on the moment of time at which the motion begins since Equation (177) does not explicitly depend on time. This means that each different trajectory in the state line does not correspond to just one motion, but to an infinite set of motions, each of which has a different t_0.

The trajectories of the representative point on the state line (the x-axis) can be points (equilibrium states), line segments (between equilibrium states), half lines (between an equilibrium state and infinity), or, finally, the whole line; when $f(x)$ does not have real roots.*

Let us study how the trajectories depend on the type of the function $f(x)$.

Let us examine the plane which has the variables $x, z = dx/dt$. If $z = f(x)$ does not intersect the x-plane, that is, if the function $f(x) = 0$ does not have real solutions, then the trajectory will be the whole state line.

Now, suppose that $f(x)$ has real roots $x = x_1, x = x_2, \ldots, x = x_k$. In this case, the curve $z = f(x)$ will intersect the x-axis several times (Figure 90).

If $f(x) < 0$, then the representative point will move in the direction of decreasing x (to the left). It follows from Figure 90 that the trajectories can be of three types: equilibrium states, segments between equilibrium states, and half lines (from an equilibrium state to infinity). It is obvious that the character of the subdivision of the state line into trajectories is completely determined by the equilibrium states.

The question of the type of the equilibrium states was already answered in Section 7.

We shall now study the possibility of periodic solutions in a degenerate system. As we have already seen, if the function $f(x)$ is an analytic function in the differential equation (177), then there will be no periodic solu-

*The representative point cannot reach an equilibrium state in a finite time because this would be a contradiction of the uniqueness theorem.

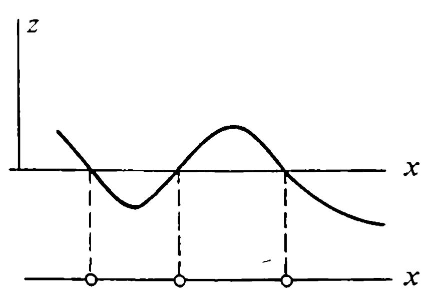

FIGURE 90

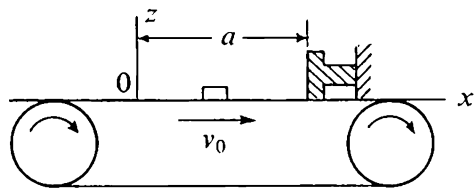

FIGURE 91

tions. Likewise, one can assert that if $f(x)$ is a single-valued function, then again there will be no periodic solutions because for continuous periodic motions the system must pass through the same value of x twice with two different values of dx/dt.

We will prove that if $f(x)$ is a single-valued function, then periodic motions are possible in the system when Equation (177) does not describe the motion of the system at all points.

The possible cases occur when Equation (177) is applicable only for some bounded region determined by the conditions of the problem or when Equation (177) demands special interpretation at some particular values of the variables.

As an example, we shall study the following mechanical system: on a corrugated belt which is moving at a constant velocity v_0 there is a point mass m, which is also moving. A rigid obstacle is placed so that the point mass will collide with it (Figure 91).

We choose the simplest idealization of friction, that is, Coulomb friction. Using the notation $\dot{x} = y$, the equation of motion becomes

$$\frac{dy}{dt} = f(y),$$

where

$$f(y) = \begin{cases} -\dfrac{T}{m} \text{ when } y > v_0, \\ 0 \text{ when } y = v_0, \\ \dfrac{T}{m} \text{ when } y < v_0. \end{cases} \tag{178}$$

Here T is the magnitude of the force of friction.

When $x = a$, Equation (178) is not valid. We shall suppose that when the point mass comes into contact with the obstacle, there is a completely elastic collision.

The problem can be solved by using the state line and the auxiliary yt-plane. But, for a more visual interpretation we shall go back to the variable x. Then, the equations of motion have the form

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -T\ (\dot{x} > v_0),\\ m\frac{d^2x}{dt^2} &= 0 \quad (\dot{x} = v_0),\\ m\frac{d^2x}{dt^2} &= T \quad (\dot{x} < v_0). \end{aligned}\right\} x < a \tag{179}$$

Again, we are supposing that when $x = a$, there is an absolutely elastic collision.

The general solution of Equation (179) has the form

$$\begin{aligned} x &= -\frac{1}{2}Ft^2 + c_1t + c_2 \quad (\dot{x} > v_0);^*\\ x &= v_0t + c_3 \qquad\qquad\quad (\dot{x} = v_0);\\ x &= \frac{1}{2}Ft^2 + c_4t + c_5 \quad\ (\dot{x} < v_0), \end{aligned}$$

where $F = T/m$.

The state plane for Equation (179) is drawn on Figure 92. When $y > v_0$, the trajectories are given by the equation

$$y^2 = -2Fx + A. \tag{180}$$

When $y > v_0$, we have

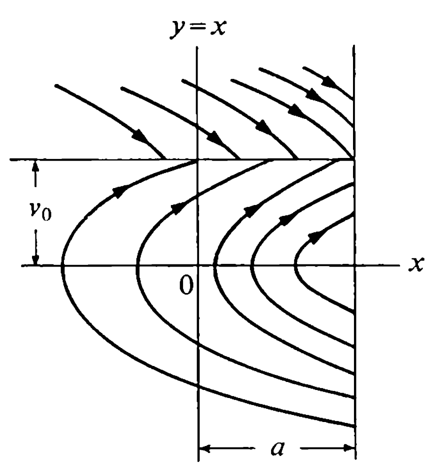

FIGURE 92

*We note that there are only two independent arbitrary constants of integration.

$$y^2 = 2Fx + B. \tag{181}$$

And, the line $y = v_0$ is a trajectory.

Let us look at the motion when the initial conditions are

$$x = x_0, \quad y = y_0 > v_0.$$

The first stage of the motion of the representative point is along the parabola (180). If when $x = a, y = v_1 \geqslant v_0$, then the representative point jumps to the point with coordinates $x = a, y = -v_1$. Afterwards, the representative point moves along the parabola (181) to the point of intersection of this parabola with the line $y = v_0$. The coordinates of the point of intersection are $x = a - v_1^2 - v_0^2/2F, y = v_0$.

After having arrived at the line $y = v_0$, the representative point moves along this line to the point with coordinates $x = a, y = v_0$. Then it jumps to the point $x = a, y = -v_0$. Then, after having traveled along the parabola (181), the representative point once again arrives at the point $x = a, y = v_0$ on the phase plane; it then jumps to the point $x = a, y = -v_0$ and begins to repeat this motion.

We will now show what the picture of the motion of the representative point means. Under the initial conditions $x = x_0 < a$, $y = y_0 > v_0$ the material point moves along the belt faster than the belt itself. In this case, the velocity will decrease, approaching the velocity v_0. When it reaches the obstacle at $x = a$, there is a completely elastic collision and the velocity, which had the value $\dot{x} = y = v_1 \geqslant v_0$ before the collision, takes on the new value $\dot{x} = -v_1$. Then, the point moves along the belt against its motion until its velocity becomes equal to the velocity of the belt. And, the next time the point strikes the obstacle it has the velocity v_0. After the collision the velocity is $\dot{x} = -v_0$ and the point moves against the motion of the belt until once again, because of the friction, it achieves the velocity v_0.

If for either of the initial conditions $y > v_0$ or $y < v_0$ the parabolas (180) and (181) intersect the line $y = v_0$, then the motion of the representative point will be periodic in the same way that it is in the case examined above.

With initial conditions under which the parabola (181) does not intersect the line $y = v_0$, the motion of the representative point will have the following form: it will move along the parabola (181) until it reaches the point $x = a$, $y = v_2 < v_0$, then it will jump to the point $x = a$, $y = -v_2$, and will begin the same motion along the parabola (181) over again, and so on.

Let us examine the same problem for the case when the obstacle is simply elastic (Figure 93).

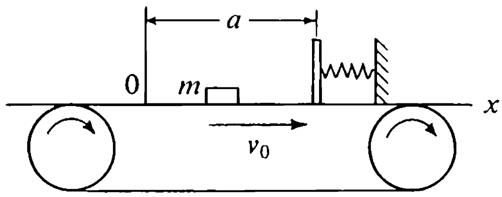

FIGURE 93

In this case, the equations have the form

$$\left.\begin{array}{l}\left.\begin{array}{l} m\dfrac{d^2x}{dt^2} = -T \quad (x \leqslant a) \\ m\dfrac{d^2x}{dt^2} = -T - c(x-a) \quad (x > a) \end{array}\right\} \dot{x} > v_0 \\ m\dfrac{d^2x}{dt^2} = 0 \left(x \leqslant a + \dfrac{T}{c}, \quad \dot{x} = v_0\right) \\ \left.\begin{array}{l} m\dfrac{d^2x}{dt^2} = T \quad (x \leqslant a) \\ m\dfrac{d^2x}{dt^2} = T - c(x-a) \quad (x > a) \end{array}\right\} \dot{x} < v_0. \end{array}\right\} \tag{182}$$

The general solution of the Equations (182) is as follows:

$$\left.\begin{array}{l}\left.\begin{array}{ll} x = -\dfrac{1}{2}Ft^2 + c_1 t + c_2 & (x \leqslant a) \\ x = c_3 \sin \omega t + c_4 \cos \omega t + a - \dfrac{T}{c} & (x > a) \end{array}\right\} \dot{x} > v_0 \\ x = v_0 t + c_5 \left(x \leqslant a + \dfrac{T}{c}, \qquad \dot{x} = v_0\right) \\ \left.\begin{array}{ll} x = \dfrac{1}{2}Ft^2 + c_6 t + c_7 & (x \leqslant a) \\ x = c_8 \sin \omega t + c_9 \cos \omega t + a + \dfrac{T}{c} & (x > a) \end{array}\right\} \dot{x} < v_0 \end{array}\right\} \tag{183*}$$

The state plane for Equation (182) is shown in Figure 94 where $c > T^2/mv_0^2$.

*There are two arbitrary independent constants: $\omega^2 = c/m$.

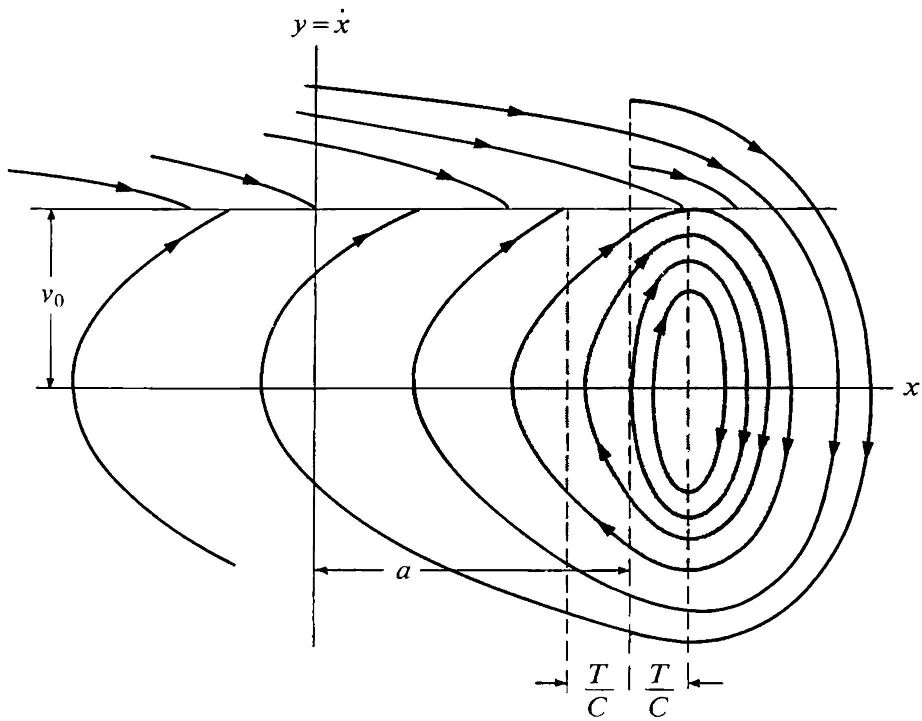

FIGURE 94

When $y > v_0$, the trajectories are described by the equations:

$$y^2 = -2Fx + A, \tag{184}$$

for $x \leqslant a$, and

$$\left[x - \left(a - \frac{T}{c}\right)\right]^2 + \frac{y^2}{\omega^2} = D. \tag{185}$$

for $x > a$.

The line $y = v_0$ is the trajectory when $x < a + T/c$. When $y < v_0$, the trajectories are described by the equations:

$$y^2 = 2Fx + B, \tag{186}$$

for $x \leqslant a$, and

$$\left[x - \left(a + \frac{T}{c}\right)\right]^2 + \frac{y^2}{\omega^2} = E \tag{187}$$

for $x > a$.

Let us study the motion under the initial conditions $x_0, y > v_0$.

The representative point will will move along the parabola (184), and there will be the following possibilities:

1. The parabola (184) intersects the line $y = v_0$ where $x < a$. Then the representative point begins motion along this line up to the point with coordinates $x = a + T/c, y = v_0$. After this, it moves along the ellipse of (187) to the point of intersection of this trajectory with the line $x = a$. The point of intersection has coordinates $x = a, y = -\sqrt{v_0^2 - (T^2/mc)}$. Then the representative point moves along the parabola (186) to the point with coordinates $x = a, y = \sqrt{v_0^2 - (T^2/mc)}$; after this, it moves along the ellipse (187) to the point $x = a + T/c, y = v_0$. Then the motion repeats itself, that is, it goes along the ellipse (187) again and so on.

2. The parabola (184) intersects the line $x = a$. Let the point of intersection have coordinates $x = a, y = v_1 > v_0$. The representative point moves along the ellipse (185).

If the trajectory intersects the line $y = v_0$ at $x < a + T/c$, then it will move along this line up to the point $x = a + T/c$, $y = v_0$, which it will begin the periodic motion described above.

If the trajectory intersects the line $y = v_0$ at the point $x = x_1, y = v_0$, where

$$x_1 = a - \frac{T}{c} + \sqrt{\frac{T^2}{c^2} + \frac{v_1^2 - v_0^2}{c} m} > a + \frac{T}{c}, \tag{188}$$

then its further motion will be along the ellipse (187).

The length of time of the motion of the representative point along the ellipse (185) is

$$t_1 = \sqrt{\frac{m}{c}} \arccos \left[\frac{v_0 v_1 + \dfrac{T}{\sqrt{mc}} \sqrt{v_1^2 - v_0^2 + \dfrac{T^2}{mc}}}{v_1^2 + \dfrac{T^2}{mc}} \right].$$

As it moves along the ellipse (185), the representative point reaches the point with coordinates

$$x = a, y = -\sqrt{v_0^2 + \left[\sqrt{\frac{T^2}{mc} + v_1^2 - v_0^2} - \frac{2T}{\sqrt{mc}}\right]^2} = y_2. \tag{190}$$

The length of time of its motion along the ellipse (187) is

$$t_2 = \sqrt{\frac{m}{c}}$$

$$\times \arccos\left[\frac{-\left(x_1 - a - \frac{T}{c}\right)T + mv_0^2\sqrt{\left(x_1 - a - \frac{T}{c}\right)^2 - \frac{T^2}{c^2} + \frac{mv_0^2}{c}}}{c\left(x_1 - a - \frac{T}{c}\right)^2 + mv_0^2}\right].$$

From the point of the state plane with coordinates $x = a$, $y = y_2$ the representative point moves along the parabola (186) until it meets the phase line $y = v_0$. As it moves along the line $y = v_0$ it reaches the point $x = a + T/c$, $y = v_0$ and then begins the motion described above.*

If the initial conditions are $x = x_0$, $y = y_0 < v_0$, then the representative point moves along the parabola (186) until it comes to the line $y = v_0$. The further motion of the representative point which has arrived at the line $y = v_0$, has already been described.

Initial conditions which correspond to the shaded region of the state plane (Figure 94), correspond to periodic motions.

If we take the limiting case for an absolutely rigid obstacle by letting $c \to \infty$, then $t_1 \to 0$, $t_2 \to 0$, since $x_1 \to a$, $y_2 \to -v_1$, and the ellipses (185) and (187) degenerate to the line $x = a$. That is, in the limit, we obtain the motions which we have already found when we used the hypothesis of an absolutely elastic collision (that is, an infinitely large collision force and an infinitesimal moment during which the velocity changes).

As a second example, we shall study the theory of mechanical relaxational waves, developed by Khaykin and Khaydanovskiy [26].

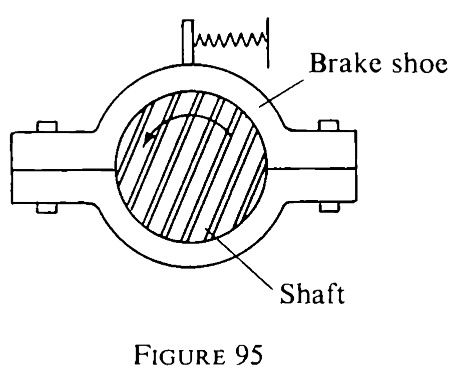

FIGURE 95

* It is possible that the representative point moving along the parabola (181) will arrive at the point $x = a, y = \sqrt{v_0^2 + [\sqrt{(T^2/mc) + v_1^2 - v_0^2} - (2T/\sqrt{mc})]^2}$. Then, moving along the ellipse (185), it will arrive at the line $y = v_0$.

A small mass in the form of a brake shoe is wrapped around a uniformly rotating shaft with a heavy friction and is fastened to a plate with a spring (Figure 95). If we assume that the mass of the brake shoe is sufficiently small, we can separate the process of motion into two basically different types:

1) The type in which the restoring force is much greater than the effect of the moment of inertia on the angular acceleration. Consequently, the motion of the system is determined by the friction force and the force of the spring; the position of the system will change considerably but the velocity comparatively little, that is, the acceleration will be small.

2) That type in which the effect of the moment of inertia on the acceleration is much greater than the restoring force and the motion is determined mainly by the effect of the moment of inertia on the acceleration. Since the moment of inertia is small, the acceleration is large, and the coordinate changes very little but the velocity changes sharply.

The equation for the motion of the brake shoe has the form (where $m = 0$)

$$r\{F[(\Omega - \dot{\varphi})\, r]\} = c\varphi, \tag{191}$$

where c is the coefficient of elasticity of the system, Ω is the angular velocity of the shaft, $\dot{\varphi}$ is the absolute angular acceleration of the brake shoe, r is the radius of the shoe, and F is a function which expresses the dependence of the force of friction on the velocity of the brake shoe relative to the shaft which it rubs against.

We choose a friction characteristic which is schematically shown in Figure 96.

For a visual representation of the motion, instead of the state line we introduce a state curve. We choose this to be the friction characteristic; this is possible since the coordinate of the shoe and the friction force are proportional to one another in those regions where there is continuous motion [see Equation (191)].

In Figure 96 the abscissa shows the relative velocity $v = r(\Omega - \dot{\varphi})$. Hence, $v = 0$ corresponds to motion of the shaft with the shoe, and $v_0 =$

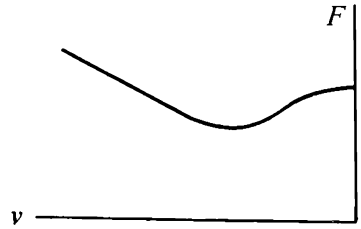

FIGURE 96

$r\Omega$ corresponds to the absence of absolute motion of the shaft, $\dot{\varphi} = 0$ is the only equilibrium state.

In studying the assumed friction characteristic we must always keep in mind the fact that when $v = 0$ the friction force can have any value between zero and F_0, which is the friction force when the shoe is at rest. This means that the graph of the friction force has a vertical branch which coincides with the coordinate axis on the segment between $F = 0$ and $F = F_0$.

The direction of motion of the representative points along the state curve is determined by the sign of $\ddot{\varphi}$.

Differentiating Equation (191) we get

$$c\dot{\varphi} = -r^2 F'[(\Omega - \dot{\varphi})r]\,\ddot{\varphi}.$$

If $F' > 0$, the signs of $\dot{\varphi}$ and $\ddot{\varphi}$ are opposite.

If $F' < 0$, the signs of $\dot{\varphi}$ and $\ddot{\varphi}$ are the same.

We are interested in the case when Ω is so small that v_0 lies in the segment where the friction force is decreasing. In this case the equilibrium state at $\dot{\varphi} = 0$ is unstable.

For sufficiently large $\dot{\varphi}$ the representative point always moves in a direction of decreasing $\dot{\varphi}$ until $\ddot{\varphi}$ changes sign.

From the equation

$$\ddot{\varphi} = -\frac{c}{r^2}\frac{\dot{\varphi}}{F'[(\Omega - \dot{\varphi})r]},$$

it is obvious that $\ddot{\varphi}$ changes sign either when $\dot{\varphi}$ changes sign or when F' changes sign. The behavior of the function $\ddot{\varphi}$ will be different according to whether $\dot{\varphi}$ or F' passes through zero. If $\ddot{\varphi}$ changes sign because $\dot{\varphi}$ changes sign, then $\ddot{\varphi}$ passes through the value zero. If $\ddot{\varphi}$ changes sign because F' changes sign, then as F' passes through the value zero, $\ddot{\varphi} = \pm\infty$.

The case when $\dot{\varphi}$ and $\ddot{\varphi}$ simultaneously pass through the value zero corresponds to an equilibrium state. The second case corresponds to the so-called "point of infinite acceleration." The representative point either approaches this point on both sides or moves away from it on both sides.

A point of stable infinite acceleration is basically different from points of equilibrium states. At such a point, the representative point cannot remain at the same place under any circumstances.

By examining the assumed friction characteristic, one can show that for $F'(\Omega) < 0$ the brake shoe will achieve a periodic motion. At first the shoe will rotate with the shaft ($\dot{\varphi} = \Omega$), the spring will stretch (the elastic force will increase), but also the force of friction will increase until it is equal to

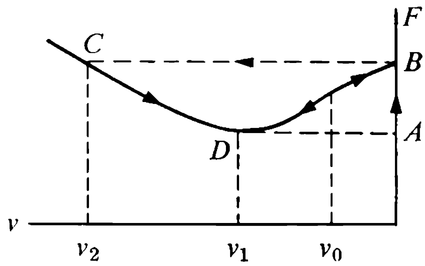

FIGURE 97

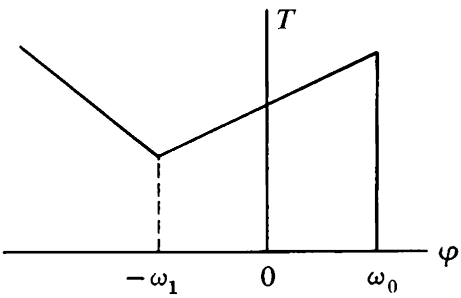

FIGURE 98

the elastic force. The representative point will move along the line segment AB of the state curve (Figure 97).

When the torque of the elastic force equals the torque of the static friction force, the representative point will reach the point B, which is the point of infinite acceleration. It can neither remain at this point nor can it move away along the state curve. One has no choice but to assume that the velocity of the brake shoe jumps from one value to another.

In order to explain where the system is after the jump, we must formulate the "jump hypothesis." In the present case the elastic force of the spring and the friction force do not change at the time of the jump (in this idealized case). Therefore, if as a "jump hypothesis" we accept the statement that the system will arrive at a new state which corresponds to the former energy of the system, then, since the coordinate of the system does not change during the jump, the jump will bring it to a position which corresponds to the former value of the force of friction, that is, to the point C. Then the representative point will move with a finite velocity and acceleration to the point D, which is a point of infinite acceleration.

Everything that was said before about the point B also relates to the point D. At the point D, another jump in the velocity will take place and the representative point will move to the point A. Then the process will begin to repeat itself.

To clarify the physical picture of this phenomenon we shall study this problem assuming that $m \neq 0$ [14].

Since our main interest is in qualitative analysis, we shall assume for our study a rough idealization of the friction characteristic illustrated in Figure 98. We shall assume that at the point where the two lines meet in the figure which illustrates the friction characteristic we have that $dT(\dot{\varphi})/dt = 0$ (this is in complete agreement with the real characteristic).

The equation of motion has the form* (Figure 95)

* $M(\dot{\varphi}) = rT(\dot{\varphi})$; r is the radius of the shaft; M_0 is the torque due to static friction.

$$J\ddot{\varphi} = -c\varphi + M(\dot{\varphi}) \begin{cases} -M_0 \leqslant M(\dot{\varphi}) \leqslant M_0, & \dot{\varphi} = \omega_0, \\ M(\dot{\varphi}) = k_1\dot{\varphi} + b_1, & -\omega_1 < \dot{\varphi} < \omega_0, \\ M(\dot{\varphi}) = k_2\dot{\varphi} + b_2, & \dot{\varphi} < -\omega_1, \end{cases} \tag{192}$$

where J is the moment of inertia of the brake shoe, M is the torque due to friction, $b_1 = M_0 - k_1\omega_0$; and $b_2 = M_0 - k_1\omega_0 - \omega_1(k_1 + k_2)$.

We choose $k_1 > k_2$. We shall suppose that $J < k_1^2/4c$.

Let us study the interval $-\omega_1 < \dot{\varphi} < \omega_0$. Equation (192) for this interval has the form

$$J\ddot{\varphi} = -c\varphi + k_1\dot{\varphi} + b_1$$

or

$$\ddot{\varphi} - 2h_1\dot{\varphi} + \omega^2\varphi = \frac{b_1}{J}, \tag{193}$$

where

$$2h_1 = \frac{k_1}{J}, \quad \omega^2 = \frac{c}{J}.$$

The equation of the state curves is

$$\left[\dot{\varphi} - \beta_1\left(\varphi - \frac{b_1}{c}\right)\right]^{\beta_1} = c_1\left[\dot{\varphi} - \beta_2\left(\varphi - \frac{b_1}{c}\right)\right]^{\beta_2}, \tag{194}$$

where

$$\beta_1 = h_1 + \sqrt{h_1^2 - \omega^2}, \ \beta_2 = h_1 - \sqrt{h_1^2 - \omega^2} \ (\beta_1 > 0, \beta_2 > 0).$$

The point $\varphi = b_1/c$, $\dot{\varphi} = 0$ is a singular point of Equation (193) and is an "unstable node."

The locus of the points at which the curves (194) have horizontal tangents is

$$\dot{\varphi} = \frac{c}{k_1}\left(\varphi - \frac{b_1}{c}\right). \tag{195}$$

The locus of the vertical tangents is $\dot{\varphi} = 0$.

We note that the line (195) passes through the point with the coordinates $\varphi = M_0/c$, $\dot{\varphi} = \omega_0$.

We shall study the region where $\dot{\varphi} < -\omega$. In this interval Equation (192) has the form

$$J\ddot{\varphi} = -c\varphi - k_2\dot{\varphi} + b_2$$

or

$$\ddot{\varphi} + 2h_2\dot{\varphi} + \omega^2\varphi = \frac{b_2}{J},$$

where $2h_2 = k_2/J$.

The equation of the trajectories is

$$\left[\dot{\varphi} - \alpha_1\left(\varphi - \frac{b_2}{c}\right)\right]^{\alpha_1} = c_2\left[\dot{\varphi} - \alpha_2\left(\varphi - \frac{b_2}{c}\right)\right]^{\alpha_2}, \tag{196}$$

where

$$\alpha_1 = -h_2 + \sqrt{h_2^2 - \omega^2}, \quad \alpha_2 = -h_2 - \sqrt{h_2^2 - \omega^2}$$
$$(\alpha_1 < 0, \alpha_2 < 0).$$

The locus of the horizontal tangents to these curves (196) is

$$\dot{\varphi} = -\frac{c}{k_2}\left(\varphi - \frac{b_2}{c}\right). \tag{197}$$

The locus of the vertical tangents is

$$\dot{\varphi} = 0.$$

The point $\varphi = b_2/c$, $\dot{\varphi} = 0$ is a stable node.

The lines (195) and (197) intersect at the point with coordinates

$$\varphi = \frac{M_0}{c} - \frac{k_1}{c}(\omega_0 + \omega_1), \quad \dot{\varphi} = -\omega_1.$$

The state plane for Equation (192) is shown in Figure 99. The direction of motion of the trajectories is indicated by the arrows.

We shall examine the motion under the initial conditions

$$-\frac{M_0}{c} < \varphi < \frac{M_0}{c}, \quad \dot{\varphi} = \omega_0.$$

The representative point will move, for example, from the point A_0 along the line to the point A_1 with coordinates $\varphi = M_0/c$, $\dot{\varphi} = \omega_0$, and this corresponds to motion of the brake shoe with the shaft. Then, the motion starts along the trajectory of the family (194) which has a horizontal tangent at the point A_1 ($\varphi = M_0/c$, $\dot{\varphi} = \omega_0$) and which intersects the axis $\dot{\varphi} = 0$ at a right angle. From the point A_2 on the line $\dot{\varphi} = -\omega_1$ the representative point moves along the trajectory of the family of curves (196) which has a horizontal tangent at its intersection with the line (197) and which intersects the line $\dot{\varphi} = -\omega_1$ at the point A_3. At this point we have

that $b_2/c < \varphi < b_1/c - k_1\omega_1/c$, since the locus of the vertical tangents for this family is the line $\dot{\varphi} = 0$.

Then, the representative point moves along the trajectory of the family (194) until this trajectory intersects the state line $\varphi = \omega_0$ at the point A_4. If $\beta_2 > c/k_1 + k_2$ then the intersection takes place at a point for which $\varphi > -M_0/c$. Subsequently, the representative point will move along the trajectory $\dot{\varphi} = \omega_0$ up to the point $\varphi = M_0/c$, $\dot{\varphi} = \omega_0$. Then it will move as is described above. Under this choice of initial conditions we get a periodic motion. We shall explain what happens to the system if $J \to 0$.

In the interval $-\omega_1 < \dot{\varphi} < \omega_0$ for Equation (192) this equation reduces to the form

$$-c\varphi + M(\dot{\varphi}) = 0$$

or

$$-c\varphi + k_1\dot{\varphi} + b_1 = 0.* \tag{198}$$

* It would be necessary to map the motion of the brake shoe described by Equation (198) onto the state line, but for the purposes of visual interpretation we map the motion onto the state curve in the usual state plane. We choose this state curve to be the broken line which is obtained from Equations (194) and (196) by letting $J \to 0$.

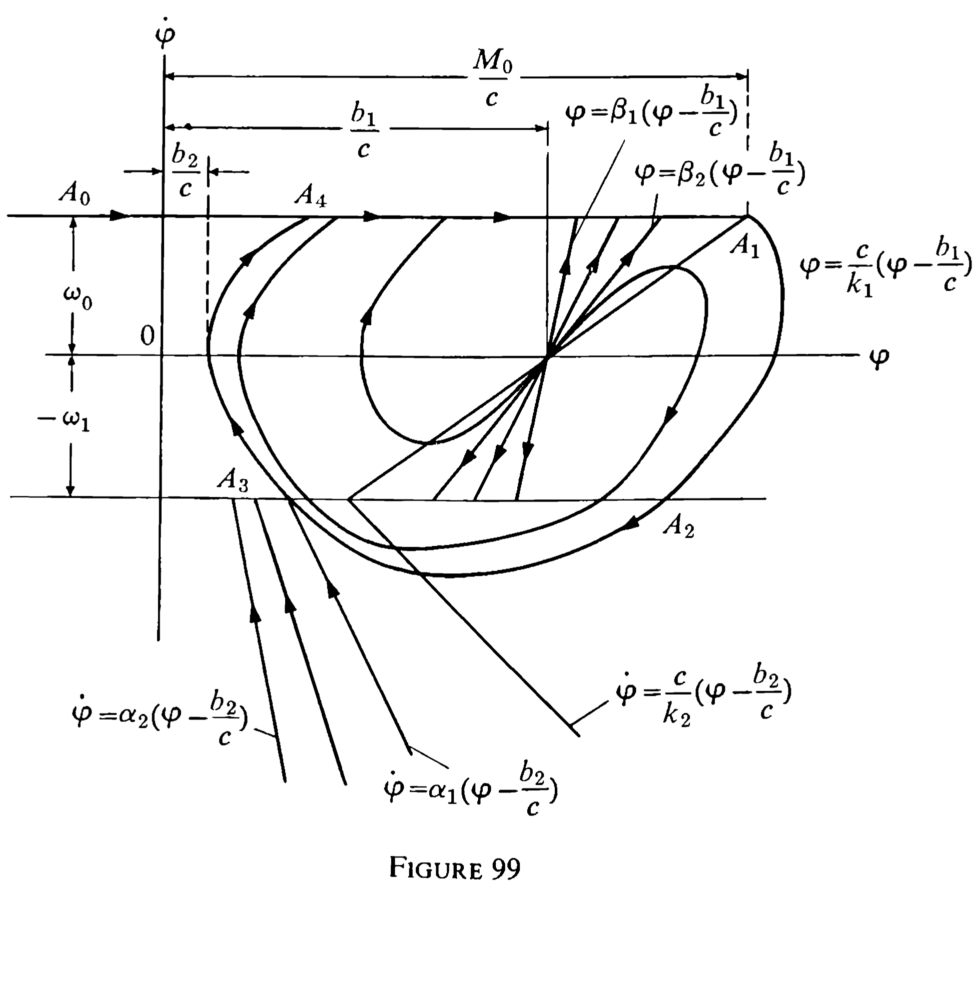

FIGURE 99

The family of trajectories reduces to the lines $\varphi = b_1/c$ and $\dot{\varphi} = c/k_1(\varphi - b_1/c)$. These are essentially the degenerate trajectories $\dot{\varphi} = \beta_1(\varphi - b_1/c)$ and $\dot{\varphi} = \beta_2(\varphi - b_1/c)$, which are obtained from (194) by putting $c_1 = \infty$ and $c_1 = 0$.

The point $\dot{\varphi} = 0$, $\varphi = b_1/c$ remains a point of unstable equilibrium.

In the region $\dot{\varphi} < -\omega_1$ Equation (192) reduces to the equation $-c\varphi + M(\dot{\varphi}) = 0$ or

$$-c\varphi - k_2\dot{\varphi} + b_2 = 0. \tag{199}$$

The family of trajectories reduces to the lines

$$\dot{\varphi} = -\frac{c}{k_2}\left(\varphi - \frac{b_2}{c}\right), \quad \varphi = \frac{b_2}{c},$$

and the point $\dot{\varphi} = 0$, $\varphi = b_2/c$ is a point of stable equilibrium. The state line $\dot{\varphi} = \omega_0$ still remains a state line. The form of the state plane is given in Figure 100.

If we let $\dot{\varphi} = \omega_0\,(-M_0/c < \varphi < M_0/c)$, then the representative point moves along the state line $\dot{\varphi} = \omega_0$ (the brake shoe is fastened to the shaft, that is, the relative speed of the shoe is zero) and arrives at the point $\varphi = M_0/c$, $\dot{\varphi} = \omega_0$. From this point it cannot move away along the state lines since the motion is directed towards this point. Also, the representative point cannot remain at this point since $\ddot{\varphi} = \infty$, that is, the point of

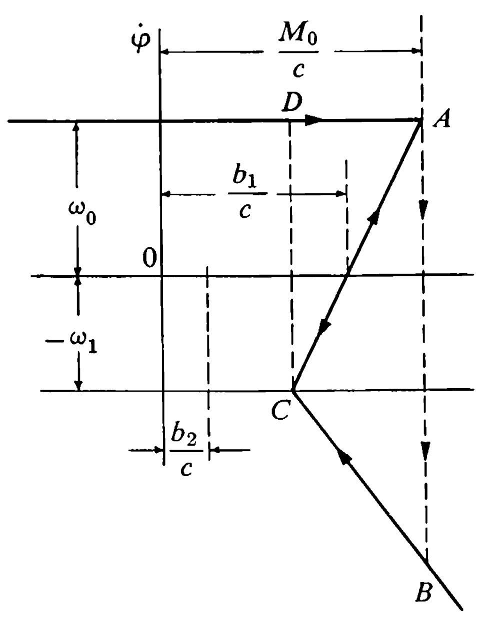

FIGURE 100

the state plane at $\varphi = M_0/c$, $\dot{\varphi} = \omega_0$ is a stable point of infinite acceleration. In fact, if we differentiate the equation

$$-c\varphi + M(\dot{\varphi}) = 0,$$

then

$$c\dot{\varphi} = \frac{dM(\dot{\varphi})}{dt}\ddot{\varphi},$$

and hence, $\ddot{\varphi} = c\dot{\varphi}/M'(\dot{\varphi})$, but at the point $\varphi = M_0/c$, $\dot{\varphi} = \omega_0$ we have $M'(\dot{\varphi}) = M'(\omega_0) = 0$.

It remains for us to assume that the velocity changes with a jump and that the jump hypothesis is such that the system keeps its former energy. Since it is accepted that the brake shoe has a moment of inertia equal to zero, the kinetic energy is zero. So all the energy of the system will be potential energy. Therefore the jump will take place with constant φ.

On the state plane the representative point will jump from the point A to the point B, and then will move along the state line $\dot{\varphi} = -(c/k_2)(\varphi - b_2/c)$, until it arrives at the point C with coordinates $\varphi = M_0/c - (k_1/c)(\omega_0 + \omega_1)$, $\dot{\varphi} = -\omega_1$. The point C is a stable point of infinite acceleration since at this point $M'(\dot{\varphi})$ is equal to zero.

The representative point will then jump from the point C to the point D (with constant φ) on the state line $\dot{\varphi} = \omega_0$. Afterwards, the motion will continue as above.

Thus we conclude that the brake shoe will undergo a periodic discontinuous oscillation for $J = 0$; such oscillations are often called *relaxation oscillations*.

In conclusion, we note that many problems which satisfy the jump hypothesis cannot be solved because it is not possible to explain with sufficient thoroughness how the neglected "small" parameter in the equations of motion affects the physical picture of the motion [2], [22]. Often certain physical parameters are considered small in formulating the equations of motion, but they nevertheless have a significant effect on the dynamics of the system. In such cases it is clear that the order of the equations of motion must be higher. Several of the coefficients of the equations of motion may be of the same order as the parameter which is considered to be small. The introduction of such a small parameter, especially when the coefficients of the higher-order derivatives are proportional to this small parameter, makes it possible to give a significantly broader and deeper physical interpretation of the motions which take place in the dynamic system.

12. A Study of a Dynamic System Which is Described by Three Differential Equations of the Third Order

In this section we shall look at the method of studying degenerate dynamic systems which are described by a system of three differential equations of the first order by using a concrete example. We shall stop to look at only several of the simpler methods. The reader can acquaint himself in more detail with contemporary methods in the works of A. A. Andronov, N. N. Bautin, A. G. Mayer, Yu. I. Neymark, A. M. Letov, A. I. Lur'e, Ya. Z. Tsypkin, et cetera.

By way of an example, we shall study the problem of the self-excited oscillations of a special stand with an autopilot mounted on it. A sketch of the stand is drawn in Figure 101.

The stand is a horizontal platform which can rotate freely around a vertical axis. An autopilot is installed on the platform with a steering mechanism. Springs are fastened to the ends of the rod A_1A_1 of the steering mechanism and their other ends are attached to the stationary points B_1B_1.

Let θ be the angle of rotation of the stand,

ξ, the displacement of the rod of the steering mechanism,

φ, the input of the servomotor which controls the motion of the rod,

$1/i$, the coupling coefficient of between the rod and the stand,

J, the moment of inertia of the stand,

b, the effective coefficient of viscosity,

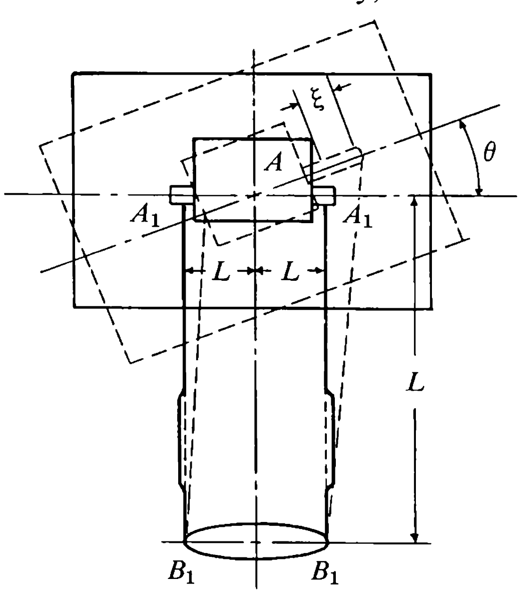

FIGURE 101

L, the lever arm of the force of the spring relative to the axis of rotation of the stand,

c, the rigidity of the spring, and

F_0, the tension of the spring at the rest position.

Then the equations for the motion of a stand with an autopilot for small θ can be expressed by the following system of equations [27], [58]:

$$\left.\begin{aligned} J\ddot{\theta} + b\dot{\theta} + 2L^2c\theta + F_0\xi &= 0, \\ \xi &= F(\varphi), \\ \varphi &= \theta - \frac{\xi}{i}, \end{aligned}\right\} \tag{200}$$

where the first equation is the equation of motion of the stand, the second is the equation of the motion of the autopilot, and the third is the feedback connection.

Eliminating the variable ξ from the system (200) and introducing dimensionless time $\tau = \omega_0 t$, where $\omega_0^2 = 2L^2C/J$, we get

$$\frac{d^3\theta}{d\tau^3} + \frac{b}{J\omega_0}\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} = -\frac{F_0}{J\omega^3}F(\varphi),$$

$$\varphi = \theta + \frac{J\omega_0^2}{iF_0}\left(\frac{d^2\theta}{d\tau^2} + \frac{b}{J\omega_0}\frac{d\theta}{d\tau} + \theta\right).$$

The characteristic of the autopilot assumed by us is drawn in Figure 102. This characteristic has two parameters: β, which determines the width of the dead zone and Q, which characterizes the maximum velocity of the motion of the rod of the steering mechanism.

By taking into consideration the form of this characteristic and by denoting

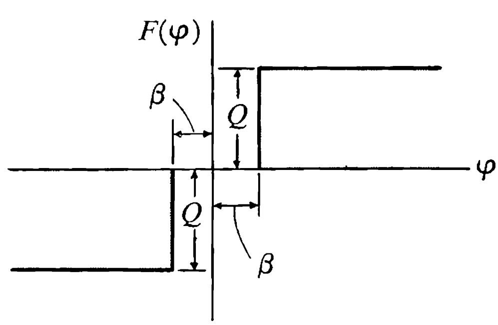

FIGURE 102

$$\frac{b}{J\omega_0} = 2h, \quad \frac{J\omega_0^2}{iF_0} = \alpha, \quad \frac{F_0 Q}{J\omega_0^3} = A, \quad L(\theta) = \frac{d^2\theta}{d\tau^2} + 2h\frac{d\theta}{d\tau} + \theta,$$

$$(h > 0,\ \alpha > 0,\ A > 0),$$

we can write the equations for the motion in the form

$$\left.\begin{aligned} &\frac{d}{d\tau}\{L(\theta)\} = -A \quad &&\text{for} \quad \theta + \alpha L(\theta) > \beta; \\ &\frac{d}{d\tau}\{L(\theta)\} = 0 \quad &&\text{for} \quad -\beta < \theta + \alpha L(\theta) < \beta; \\ \text{and} \quad &\frac{d}{d\tau}\{L(\theta)\} = A \quad &&\text{for} \quad \theta + \alpha L(\theta) < -\beta. \end{aligned}\right\} \tag{201}$$

In order to get a clear idea of the character of the motion which is described by this system of equations, we introduce the new variables

$$z = \varphi = \alpha\frac{d^2\theta}{d\tau^2} + 2h\alpha\frac{d\theta}{d\tau} + (1+\alpha)\theta,$$

$$y = \frac{d\theta}{d\tau},$$

$$x = \theta,$$

and study the motion in xyz-space.

In terms of the new variables, the system (201) has the form

$$\left.\begin{aligned} &\frac{dz}{d\tau} = y - \alpha A, \\ &\frac{dy}{d\tau} = \frac{1}{\alpha}[z - 2h\alpha y - (1+\alpha)x], \\ &\frac{dx}{d\tau} = y, \end{aligned}\right\} \quad z > \beta \tag{202}$$

$$\left.\begin{aligned} &\frac{dz}{d\tau} = y, \\ &\frac{dy}{d\tau} = \frac{1}{\alpha}[z - 2h\alpha y - (1+\alpha)x], \\ &\frac{dx}{d\tau} = y, \end{aligned}\right\} \quad -\beta < z < \beta \tag{203}$$

$$\left.\begin{aligned} \frac{dz}{d\tau} &= y + \alpha A, \\ \frac{dy}{d\tau} &= \frac{1}{\alpha}[z - 2h\alpha y - (1 + \alpha)x], \\ \frac{dx}{d\tau} &= y. \end{aligned}\right\} \quad z < -\beta \tag{204}$$

Geometrically, this means that the xyz-space is divided by the planes $z = \beta, z = -\beta$ into three parts, in each of which the motion is subject to the corresponding system of equations, the right-hand sides of which satisfy the conditions of Cauchy's theorem on the existence and uniqueness of the solution. Thus, in the region above the plane $z = \beta$ the system (202) is valid, in the region between the planes $z = \beta$ and $z = -\beta$ the system (203) holds, and in the region below the plane $z = -\beta$ the system (204) holds.

The trajectories of the first of these systems are tangent to the plane $z = \beta$ along the line $z = \beta, y = \alpha A$ (and only along this line) and for values of $y > \alpha A$ which are close to points of the plane $z = \beta$, if τ is increasing, the representative point moves in the direction of increasing z, since $dz/d\tau > 0$, and for values of $y < \alpha A$ it moves in the direction of decreasing z since $dz/d\tau < 0$.

From the solution of the system* (202) it is clear that with motion of the representative point located at the initial moment on the plane $z = \beta$ with an initial value $y_0 > \alpha A$, we have that for increasing τ, z at first increases and, then, after some interval of time, begins to decrease; finally, the representative point intersects the plane $z = \beta$ once again.

The trajectories of the system (203) which are valid in the region between the planes $z = \beta$ and $z = -\beta$, are tangent to the planes $z = \beta$ and $z = -\beta$ along the lines $z = \beta, y = 0$ and $z = -\beta, y = 0$ respectively (and only along these lines), and for points which are near to the planes $z = \beta$

---

* We have the following expression for z:

$$z = e^{-ht}[-(hc_1 + c_2\omega)\cos\omega\tau + (-hc_2 + c_1\omega)\sin\omega\tau] - (1 + \alpha)A\tau + c_3.$$

where $\omega^2 = 1 - h^2$, and c_1, c_2, and c_3 are constants of integration.

For the initial conditions $\tau = 0\, x = 0, y = y_0, z = \beta$, we get

$$c_1 = y_0 + A, \quad c_2 = \frac{h}{\omega}(A - y_0) - \frac{1 + \alpha}{\omega\alpha}x_0 + \frac{\beta}{\alpha\omega},$$

$$c_3 = 2hA - \frac{1 + \alpha}{\alpha}(x_0 - \beta).$$

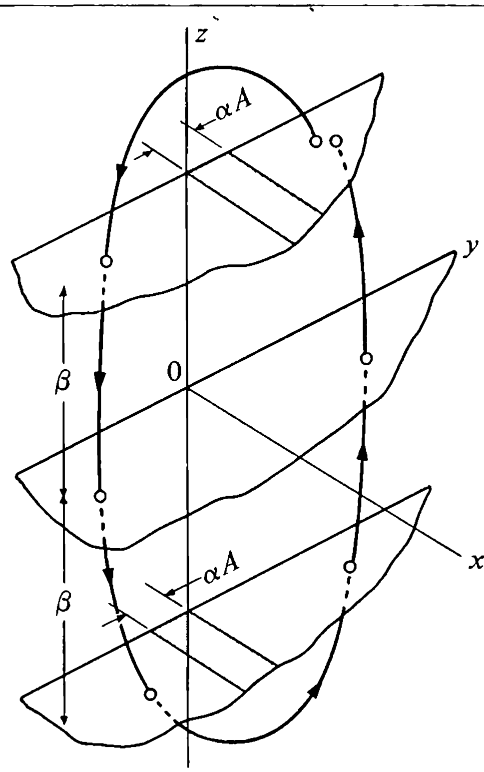

FIGURE 103

and $z = -\beta$, and which correspond to values of $y < 0$, for increasing time τ the coordinate z decreases, and for points which are close to these planes which correspond to values of $y > 0$, z increases with increasing τ. We note that the line $y = 0, z = (1 + \alpha)x$ is a line of an equilibrium state for the trajectories of the system and that not all the trajectories which intersect the plane $z = \beta$ with values $y < 0$, intersect the plane $z = -\beta$, as τ increases, but only those trajectories which pass through points for which the values y are sufficiently large in absolute value.

It is obvious that the motion of the trajectories in the region where $z < -\beta$ is completely symmetrical to their motion in the region $z > \beta$, since the system of equations (204) can be obtained from the system (202) by substituting $-x$ for x, $-y$ for y, and $-z$ for z.

By virtue of what has already been said it is clear that the strip on the plane $z = \beta$ between the lines $z = \beta, y = \alpha A$ and $z = \beta, y = 0$ and the strip on the plane $z = -\beta$ between the lines $z = -\beta, y = -\alpha A$ and $z = -\beta, y = 0$ are strips where the phase trajectories are joined* (Figure 103).

In order to solve the problem of the location of the periodic region we could apply the method of exact transformations, but this would involve

* Here, we are not touching on the question of the so-called "sliding regime."

very cumbersome calculations (although it would give a complete solution of the problem). We shall limit ourselves to a study of symmetric periodic motion by using the method of broken-line approximation.

When $\tau = 0$, let $y = y_0$ ($y_0 > \alpha A$), $z = z_0 = \beta$. Due to the character of the trajectories in the xyz-space, the representative point, which has the initial conditions already given at the time $\tau = 0$, will at some time $\tau = T_0$ arrive at the plane $z = \beta$ once again at a point with coordinates $x = x_1$, $y = y_1$, $z = z_1 = \beta$. The further motion of the representative point after an interval of time ϑ_0 will bring it to the plane $z = -\beta$, at the point with coordinates $x = x_2$, $y = y_2$, $z = z_2$.

It is not difficult to see that if in this instance $x_2 = -x_0$, $y_2 = -y_0$, $z = z_2 = -\beta$, then there will be a periodic motion which we will call a symmetrical periodic motion.

Given this assumption it follows, likewise, that the representative point will be located under the plane $z = -\beta$ after an interval of time T_0, and after an interval of time ϑ_0 it will move between the planes $z = -\beta$ and $z = \beta$. Therefore, the period of this motion is equal to

$$T = 2(T_0 + \vartheta_0).$$

This geometrical study of the symmetrical solution enables us to write equations which will determine the quantities x_0 y_0, T_0, ϑ_0. We shall find these equations by using a device which makes it possible for us to find at the same time the Fourier expansion of the desired periodic solution.

We denote by $\bar{\theta}$ the desired symmetrical periodic solution. For such a solution the following equation is valid (under the particular choice of the initial moment of time stated above):

$$\frac{d}{d\tau}\{L(\bar{\theta})\} = f(\tau),$$

where $f(\tau)$ is the function which is drawn in Figure 104.

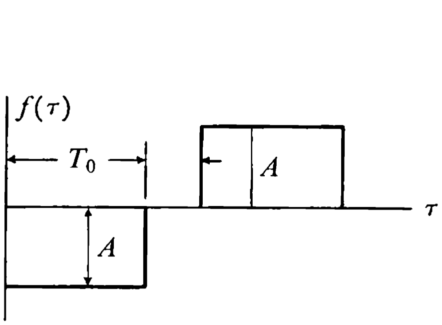

FIGURE 104

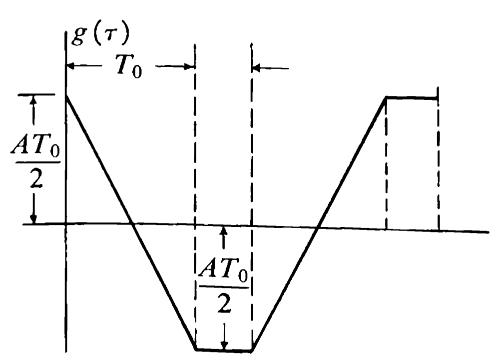

FIGURE 105

From the equation above it follows that*

$$L(\bar{\theta}) = g(\tau), \tag{205}$$

where

$$g(\tau) = -A\tau + \frac{AT_0}{2} \text{ when } 0 \leqslant \tau \leqslant T_0,$$

$$g(\tau) = -\frac{T_0 A}{2} \quad \text{when} \quad T_0 \leqslant \tau \leqslant \frac{T}{2} = T_0 + \vartheta_0,$$

and so on (Figure 105).

Hence, we get the following conditions for the quantities which characterize $\bar{\theta}$ (See (201)):

$$\{\bar{\theta}\}_{\tau=0} = \beta - \frac{\alpha A T_0}{2}, \quad \left\{\frac{d\bar{\theta}}{d\tau}\right\}_{\tau=0} = \dot{\theta}_0; \tag{206}$$

$$\{\bar{\theta}\}_{\tau=T_0} = \beta + \frac{\alpha A T_0}{2}, \quad \left\{\frac{d\bar{\theta}}{d\tau}\right\}_{\tau=T_0} = \dot{\theta}_1; \tag{207}$$

$$\{\bar{\theta}\}_{\tau=T_0+\vartheta_0} = -\beta + \frac{\alpha A T_0}{2}, \quad \left\{\frac{d\bar{\theta}}{d\tau}\right\}_{\tau=T_0+\vartheta_0} = -\dot{\theta}_0. \tag{208}$$

The solution of Equation (205) in the interval $0 < \tau \leqslant T_0$ will be

$$\bar{\theta} = e^{-h\tau}(c_1 \cos \omega\tau + c_2 \sin \omega\tau) - A\tau + \frac{AT_0}{2} + 2hA,$$

where $\omega = \sqrt{1 - h^2}$.

For the initial conditions (206), we get

$$\bar{\theta} = e^{-h\tau}\left\{\left[\beta - (1+\alpha)\frac{AT_0}{2} - 2hA\right]\cos \omega\tau \right.$$

$$\left. + \frac{1}{\omega}\left[\dot{\theta}_0 + h\beta - h(1+\alpha)\frac{AT_0}{2} + (1 - 2h^2)A\right]\sin \omega\tau\right\}$$

$$- A\tau + \frac{AT_0}{2} + 2hA;$$

$$\frac{d\bar{\theta}}{d\tau} = e^{-h\tau}\left\{(\dot{\theta}_0 + A)\cos \omega\tau + \frac{1}{\omega}\left[-h\dot{\theta}_0 - \beta \right.\right.$$

$$\left.\left. + (1+\alpha)\frac{AT_0'}{2} - h(1 - 2A)\right]\sin \omega\tau\right\} - A.$$

*After integrating we get $g(\tau) = -A\tau + c$, but since the solution is symmetric, it follows that, when $\tau = T_0/2$ and hence, $g(\tau) = -A\tau + c$, $g(T_0/2) = 0$.

For the conditions (207), we have

$$\beta + \frac{\alpha A T_0}{2} = \left[\beta - (1+\alpha)\frac{A T_0}{2} - 2hA\right]\xi \tag{209}$$

$$+ \frac{1}{\omega}\left[\dot{\theta}_0 + h\beta - h(1+\alpha)\frac{A T_0}{2} + (1-2h^2)A\right]\eta - \frac{A T_0}{2} + 2hA;$$

$$\dot{\theta}_1 = (\dot{\theta}_0 + A)\xi +$$

$$+ \frac{1}{\omega}\left[-h\dot{\theta}_0 - \beta + (1+\alpha)\frac{A T_0}{2} - h(1-2A)\right]\eta - A,$$

$$\xi = e^{-hT_0}\cos\omega T_0, \quad \eta = e^{-hT_0}\sin\omega T_0.$$

In the interval $T_0 < \tau < T_0 + \vartheta_0$ the solution of the Equation (205) has the form

$$\bar{\theta} = e^{-h\tau}(c_3\cos\omega\tau + c_4\sin\omega\tau) - \frac{A T_0}{2}.$$

For the initial conditions (207), which can be written in the form

$$\{\bar{\theta}\}_{\tau=0} = \beta + \frac{\alpha A T_0}{2}, \quad \left\{\frac{d\bar{\theta}}{d\tau}\right\}_{\tau=0} = \dot{\theta}_1,$$

we have

$$\bar{\theta} = e^{-h\tau}\left\{\left[\beta + (1+\alpha)\frac{A T_0}{2}\right]\cos\omega\tau\right.$$

$$\left. + \frac{1}{\omega}\left[\dot{\theta}_1 + h\beta + h(1+\alpha)\frac{A T_0}{2}\right]\sin\omega\tau\right\} - \frac{A T_0}{2},$$

$$\frac{d\bar{\theta}}{d\tau} = e^{-h\tau}\left\{\dot{\theta}_1\cos\omega\tau - \frac{1}{\omega}\left[h\dot{\theta}_1 + \beta + (1+\alpha)\frac{A T_0}{2}\right]\sin\omega\tau\right\}.$$

For the conditions (208) written in the form

$$\{\bar{\theta}\}_{\tau=\vartheta_0} = -\beta + \frac{\alpha A T_0}{2}, \quad \left\{\frac{d\bar{\theta}}{d\tau}\right\}_{\tau=\vartheta_0} = -\dot{\theta}_0,$$

we get

$$\left.\begin{aligned} -\beta + \frac{\alpha A T_0}{2} &= \left[\beta + (1+\alpha)\frac{A T_0}{2}\right]u + \\ &+ \frac{1}{\omega}\left[\dot{\theta}_1 + h\beta + h(1+\alpha)\frac{A T_0}{2}\right]v - \frac{A T_0}{2}, \\ -\dot{\theta}_0 &= \dot{\theta}_1 u - \frac{1}{\omega}\left[h\dot{\theta}_1 + \beta + (1+\alpha)\frac{A T_0}{2}\right]v, \end{aligned}\right\} \tag{210}$$

where

$$u = e^{-h\vartheta_0}\cos\omega\vartheta_0, \quad v = e^{-h\vartheta_0}\sin\omega\vartheta_0.$$

Eliminating the quantities $\dot{\theta}_0$ and $\dot{\theta}_1$ from Equations (209) and (210), we obtain the following equations for determining the quantities T_0 and ϑ_0

$$\left.\begin{aligned}
&\omega\left[v\left(\xi^2+\eta^2+\xi\right)-(1-u)\,\eta\right]\frac{AT_0}{2}(1+\alpha) \\
&\qquad = v\omega\,(\beta-2hA)\,(\xi^2+\eta^2-\xi)+[Av\,(1-2h^2)- \\
&\qquad\qquad -\beta\omega\,(1+u)]\,\eta; \\
&\omega\left[\eta\,(u-u^2-v^2)+(1+\xi)\,v\right]\frac{AT_0}{2}(1+\alpha) \\
&\qquad = \beta\omega\eta\,(u^2+v^2+u)+v\omega\,(\beta-2hA)\,(\xi-1)+ \\
&\qquad\qquad + Av\eta\,(1-2h^2).
\end{aligned}\right\} \tag{211}$$

These equations can be solved numerically.

We shall limit ourselves to the case in which h and A are small. In this case, T_0 and ϑ_0 can be obtained in the form of a power series in h, assuming that $A = 2hA_0$.

Writing

$$\left.\begin{aligned}
\vartheta_0 &= \tau_1 + \mu_1 h + \mu_2 h^2 + \cdots \\
T_0 &= \pi - \tau_1 + \nu_1 h + \nu_2 h^2 + \cdots
\end{aligned}\right\} \tag{212}$$

and substituting (211) in the equations, we find that

$$\sin\tau_1 = \frac{\pi\beta}{2A_0}, \quad \mu_1 = 0, \quad \nu_1 = 0, \tag{213}$$

$$\mu_2 = \frac{\pi^2(\pi-\tau_1)(1+\alpha)[3\sin^2\tau_1+(\pi-\tau_1)\sin\tau_1\cos\tau_1-(\pi-2\tau_1)\sin\tau_1]}{\Delta}$$

$$-\frac{\frac{\pi^3}{2}(\pi-\tau_1)^2(1+\alpha)^2(1-\cos\tau_1)-\sin^2\tau_1\left(-\frac{1}{3}\pi^3+\pi^2\tau_1-\pi\tau_1^2-8\pi\right)}{\Delta}$$

$$+\frac{\sin^2\tau_1\cos\tau_1\left(\frac{2}{3}\pi^3-2\pi^2\tau_1+\tau_1{}^2\pi-8\pi\right)}{\Delta}$$

$$+\frac{5\pi(\pi-\tau_1)\sin^3\tau_1-\pi\tau_1\sin\tau_1(1+\cos\tau_1)}{\Delta};$$

$$\nu_2=\frac{\frac{\pi^3}{2}(\pi-\tau_1)^2(1+\alpha)^2(1-\cos\tau_1)}{\Delta}+$$

$$+\frac{\pi^2(\pi-\tau_1)(1+\alpha)[(\pi-2\tau_1)\sin\tau_1-3\sin^2\tau_1+\tau_1\cos\tau_1\sin\tau_1]}{\Delta}$$

$$+\frac{\sin^2\tau_1(1+\cos\tau_1)\left(-\frac{1}{3}\pi^3-\pi\tau_1^2+8\pi\right)}{\Delta}$$

$$-\frac{\pi\tau_1(\pi-2\tau_1)\sin^2\tau_1+\sin\tau_1(5\pi^2-\pi\tau_1)-5\pi\tau_1\sin^3\tau_1}{\Delta}$$

$$+\frac{\sin\tau_1\cos\tau_1(3\pi^2-11\pi\tau_1)}{\Delta},$$

where

$$\Delta=-2\pi(1+\cos\tau_1)\cos\tau_1\sin\tau_1.$$

We shall now find the Fourier expansion of the symmetric solution. By expanding $g(\tau)$, we have

$$g(\tau)=\sum_{m=0}^{\infty}(M_m\cos\omega_m\tau-N_m\sin\omega_m\tau),$$

where

$$M_m=\frac{AT}{\pi^2}\frac{1}{(2m+1)^2}\left[1-\cos 2\pi(2m+1)\frac{T_0}{T}\right],$$

$$N_m=\frac{AT}{\pi^2}\frac{1}{(2m+1)^2}\sin 2\pi(2m+1)\frac{T_0}{T},$$

$$\omega_m=\frac{2(2m+1)\pi}{T}.$$

If we consider the problem of finding $\bar{\theta}$ as if it were an ordinary problem of forced oscillations, we have

$$\bar{\theta} = \sum_{m=0}^{\infty} (c_1^m \cos \omega_m \tau + c_2^m \sin \omega_m \tau),$$

where

$$c_1^m = \frac{M_m(1 - \omega_m^2) + 2N_m h \omega_m}{(1 - \omega_m^2)^2 + 4h^2\omega_m^2},$$

$$c_2^m = \frac{2M_m h \omega_m - (1 - \omega_m^2) N_m}{(1 - \omega_m^2)^2 + 4h^2\omega_m^2}.$$

We shall find the amplitude of the first harmonic. The square of this amplitude is equal to

$$k^2 = (c_1^0)^2 + (c_2^0)^2 = 2\left(\frac{AT}{\pi^2}\right)^2 \frac{1 - \cos \omega_0 T_0}{(1 - \omega_0^2)^2 + 4h^2\omega_0^2},$$

where

$$T = 2(T_0 + \vartheta_0), \quad \omega_0 = \frac{\pi}{T_0 + \vartheta_0}.$$

We shall find the value of k^2 for small h and small A ($A = 2hA_0$). In this case, neglecting the terms with h^2 from (212) and (213) we have

$$T_0 + \vartheta_0 = \pi, \quad T = 2\pi, \quad \omega_0 = 1$$

and, consequently,

$$k^2 = \frac{8A_0^2}{\pi^2}(1 - \cos T_0),$$

but $\cos T_0 = \cos(\pi - \tau_1) = -\cos \tau_1$, and, since $\sin \tau_1 = \pi\beta/2A_0$, it follows that

$$k^2 = \frac{8A_0^2}{\pi^2}\left(1 + \sqrt{1 - \frac{\pi^2\beta^2}{4A_0^2}}\right). \tag{214}$$

We note that for $\beta = 0$, $\vartheta_0 = 0$, a transcendental equation, which has a comparatively simple form for the determination of T, and which is independent of A, is

$$\frac{\omega T}{2}(1 + \alpha) = 4\,\frac{(1 - 2h^2)e^{-\frac{hT}{2}} \sin \frac{\omega T}{2} + h\omega(1 - e^{-hT})}{1 + 2e^{-\frac{hT}{2}} \cos \frac{\omega T}{2} + e^{-hT}}.$$

For sufficiently small h the expression for the period has the form

$$T = 2\pi + \left[5\pi - \frac{1}{2}(1 + \alpha)\pi^3\right]h^2 + \cdots,$$

and the amplitude of the first harmonic is $k = 4A_0/\pi$.

We will not touch on the proof of the stability of the symmetric periodic solution, since, when $\beta \neq 0$, it presents considerable practical difficulties.

We shall study this problem by applying the approximate method of slowly varying parameters. We make the suppositions that h and A are small, that is, we assume that the dimensionless quantities satisfy

$$2h = \frac{b}{J\omega_0} \ll 1, \quad A = \frac{F_0 Q}{J\omega_0^3} \ll 1$$

and do not have the same order in a neighborhood of zero.

By way of a small parameter which characterizes the nearness of the system in question to a linear conservative one, we take the parameter $\mu = b/J\omega_0$ and then, denoting $A_0 = F_0 Q/\omega_0^2 b$, we can write the equations of motion in the form

$$\left.\begin{aligned} \frac{d^3\theta}{d\tau^3} + \frac{d\theta}{d\tau} &= -\mu\left(\frac{d^2\theta}{d\tau^2} + A_0\frac{F(\varphi)}{Q}\right), \\ \varphi &= \theta + \alpha\left(\frac{d^2\theta}{d\tau^2} + \mu\frac{d\theta}{d\tau} + \theta\right). \end{aligned}\right\} \tag{215}$$

By eliminating φ from the Equations (215) and neglecting the terms with μ^2, we conclude

$$\frac{d^3\theta}{d\tau^3} + \frac{d\theta}{d\tau} = \mu\left\{-\frac{d^2\theta}{d\tau^2} - \frac{A_0}{Q}F\left[\alpha\frac{d^2\theta}{d^2\tau} + (1 + \alpha)\theta\right]\right\}.$$

In this way we see that the equations of motion are of the type

$$\frac{d^3\theta}{d\tau^3} + \frac{d\theta}{d\tau} = \mu f\left(\theta, \frac{d\theta}{d\tau}, \frac{d^2\theta}{d\tau^2}\right).$$

If we are looking for a solution of this equation in the form

$$\theta = a\cos\tau + b\sin\tau + c,$$

and assuming that a, b, and c are slowly changing functions of τ with respect to time, then in a manner analogous to the technique used in Section 7 we can show that the following equations are approximately valid for the quantities a, b, and c:

$$\left.\begin{aligned}\frac{da}{d\tau} &= \mu\psi_1(a,b,c),\\ \frac{db}{d\tau} &= \mu\psi_2(a,b,c),\\ \frac{dc}{d\tau} &= \mu\psi_3(a,b,c),\end{aligned}\right\} \tag{216}$$

where

$$\psi_1 = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau + c, -a\sin\tau + b\cos\tau,$$
$$-a\cos\tau - b\sin\tau)\cos\tau\, d\tau;$$

$$\psi_2 = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau + c, -a\sin\tau + b\cos\tau,$$
$$-a\cos\tau - b\sin\tau)\sin\tau\, d\tau;$$

$$\psi_3 = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau + b\sin\tau + c, -a\sin\tau + b\cos\tau,$$
$$-a\cos\tau - b\sin\tau)\, d\tau.$$

Letting $k^2 = a^2 + b^2$ in our case we get the following system:

$$\left.\begin{aligned}\frac{dk^2}{d\tau} &= -k^2 + \frac{A_0 k}{\pi Q}\int_0^{2\pi} F[k\cos\xi + (1+\alpha)c]\cos\xi\, d\xi,\\ \frac{dc}{d\tau} &= -\frac{A_0}{2\pi Q}\int_0^{2\pi} F[k\cos\xi + (1+\alpha)c]\, d\xi.\end{aligned}\right\} \tag{217}$$

The characteristics of $F(\varphi)$, which we have assumed, make it simple to calculate the integrals in Equation (217).

We shall study only the first quadrant of the plane in the variables kc, since $k = \sqrt{a^2 + b^2} > 0$, and, when $c < 0$, we get a symmetrical graph.

The first quadrant of the kc-plane is divided into the following six regions in which the right sides of Equations (217) have different analytic expressions.

1. $-k + (1 + \alpha)c > \beta$. In this region Equations (217) have the form

$$\frac{dk}{d\tau} = -\frac{k}{2}, \quad \frac{dc}{d\tau} = -A_0.$$

All the other regions are located in that part of the kc-plane for which $-k + (1 + \alpha)c < \beta$.

2. $k + (1 + \alpha)c < \beta$. Equations (217) have the form

$$\frac{dk}{d\tau} = -\frac{k}{2}, \quad \frac{dc}{d\tau} = 0.$$

The resulting equations have line segments of equilibrium states on the c-axis. This family of equilibrium states can be found directly (without using the method of slowly varying parameters) by analyzing the system of linear equations with constant coefficients which are valid when $|\varphi| < \beta$.

3. $k + (1 + \alpha)c > \beta$, $k - (1 + \alpha)c < \beta$, $(1 + \alpha)c < \beta$. Equations (217) take the form

$$\frac{dk}{d\tau} = -\frac{k}{2} + \frac{A_0}{\pi}\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2},$$

$$\frac{dc}{d\tau} = -\frac{A_0}{\pi}\arcsin\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2}.$$

4. $k + (1 + \alpha)c > \beta$, $k - (1 + \alpha)c > \beta$, $(1 + \alpha)c < \beta$. Equations (217) can be written in the form

$$\frac{dk}{d\tau} = -\frac{k}{2} + \frac{A_0}{\pi}\left\{\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2} + \sqrt{1 - \left[\frac{\beta + (1 + \alpha)c}{k}\right]^2}\right\},$$

$$\frac{dc}{d\tau} = -\frac{A_0}{\pi}\left\{\arcsin\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2} - \arcsin\sqrt{1 - \left[\frac{\beta + (1 + \alpha)c}{k}\right]^2}\right\}.$$

5. $k - (1 + \alpha)c < \beta$, $k + (1 + \alpha)c > \beta$, $(1 + \alpha)c > \beta$. Equations (217) will be

$$\frac{dk}{d\tau} = -\frac{k}{2} + \frac{A_0}{\pi}\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2},$$

$$\frac{dc}{d\tau} = -\frac{A_0}{\pi}\left\{\pi - \arcsin\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2}\right\}.$$

6. $k - (1 + \alpha)c > \beta$, $k + (1 + \alpha)c > \beta$, $(1 + \alpha)c > \beta$. Equations (217) will be

$$\frac{dk}{d\tau} = -\frac{k}{2} + \frac{A_0}{\pi}\left\{\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2} + \sqrt{1 - \left[\frac{\beta + (1 + \alpha)c}{k}\right]^2}\right\},$$

$$\frac{dc}{d\tau} = -\frac{A_0}{\pi}\left\{\pi - \arcsin\sqrt{1 - \left[\frac{\beta - (1 + \alpha)c}{k}\right]^2} - \arcsin\sqrt{1 - \left[\frac{\beta + (1 + \alpha)c}{k}\right]^2}\right\}.$$

The boundaries of these regions are shown in Figure 106.

From an examination of these equations it follows that the singular points can be only in the fourth region when $c = 0$ (on the k-axis) and the line segment of equilibrium states will always be on the c-axis (region II).

Setting the right parts of the equations equal to zero for region VI and letting $c = 0$, we get an equation which determines those k, which correspond to singular points

$$-\frac{k}{2} + \frac{2A_0}{\pi}\sqrt{1 - \left(\frac{\beta}{k}\right)^2} = 0$$

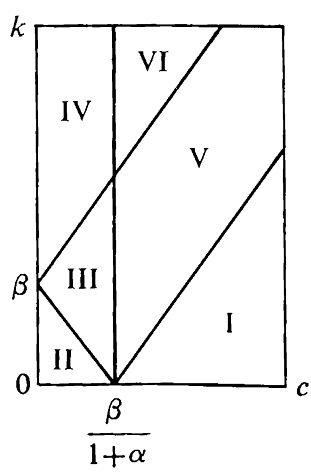

FIGURE 106

or

$$k^4 - \frac{16A_0^2}{\pi^2}k^2 + \frac{16A_0^2}{\pi^2}\beta^2 = 0. \tag{218}$$

Thus, the singular points P_1 and P_2, which correspond to periodic motion in the resulting system, exist when $A_0 > (\pi/2)\,\beta$ and have the coordinates

$$\left.\begin{aligned} k_1^2 &= \frac{8A_0^2}{\pi^2}\left(1 + \sqrt{1 - \frac{\pi^2\beta^2}{4A_0^2}}\right), \quad c_1 = 0, \\ k_2^2 &= \frac{8A_0^2}{\pi^2}\left(1 - \sqrt{1 - \frac{\pi^2\beta^2}{4A_0^2}}\right), \quad c_2 = 0. \end{aligned}\right\} \tag{219}$$

For these points the quantities p and q, which determine their character and stability (Section 3), have the form

$$p = \frac{8A_0^2(1+\alpha)}{\pi^2k^2} - \left(-\frac{1}{2} + \frac{8A_0^2\beta^2}{\pi^2k^4}\right),$$

$$q = -\frac{8A_0^2(1+\alpha)}{\pi^2k^2}\left(-\frac{1}{2} + \frac{8A_0^2\beta^2}{\pi^2k^4}\right).$$

When $A_0 > (\pi/2)\,\beta$, for the point P_1 we have $p > 0$ and $q > 0$, that is, the singular point will be a stable node; for the point P_2 we have $q < 0$, that is, the point P_2 will be a saddle point.

The dependence of the ordinates of the equilibrium states at P_1 and P_2 on the parameters F_0 and β can be expressed with the help of the "amplitude curve,"

$$k^4 - a_0^2F_0^2k^2 + a_0^2F_0^2\beta^2 = 0,$$

where $a_0 = 4Q/\pi b\omega_0^2$ (see Equation (218)).

An amplitude curve for a fixed β is drawn in Figure 107, where the stable portion is indicated by the thicker lines.

For sufficiently large k and c we have that

$$\frac{dk}{d\tau} < 0, \quad \frac{dc}{d\tau} < 0.$$

If we know the behavior of the integral curves at infinity, the distribution and character of the equilibrium states as well as the directions at which the integral curves intersect the boundaries of the regions, we can give a complete picture of the subdivision of the kc-plane into trajectories.

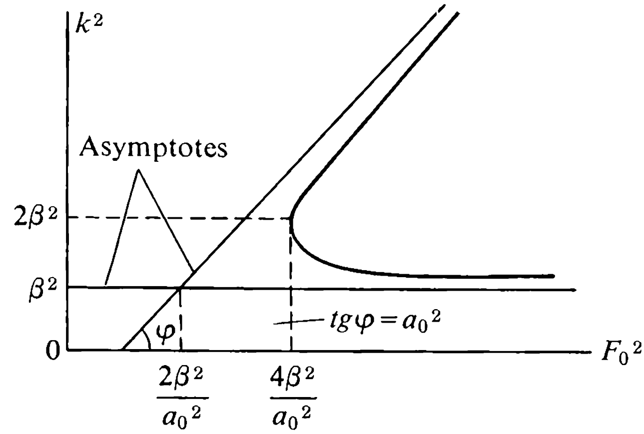

FIGURE 107

Figure 108 gives pictures of the state plane for various values of F_0^2. It is obvious that if $F_0^2 > 4\beta^2/a_0^2$, there can be stable periodic oscillations in the system, but they can be realized only if the system has initial conditions which correspond to points of the kc-plane which lie inside the region of "stability in the large" for the point P_1.

If the system is not "thrown" into this region, then it will converge to one of the equilibrium states on the c-axis.

When $\beta = 0$, we have a stable periodic solution with an amplitude of

$$k = \frac{4A_0}{\pi}.$$

The region of "stability in the large" for this solution will now be the whole kc-plane. The amplitude of this solution increases linearly with F_0.

If we compare the results obtained by direct analysis of the symmetric

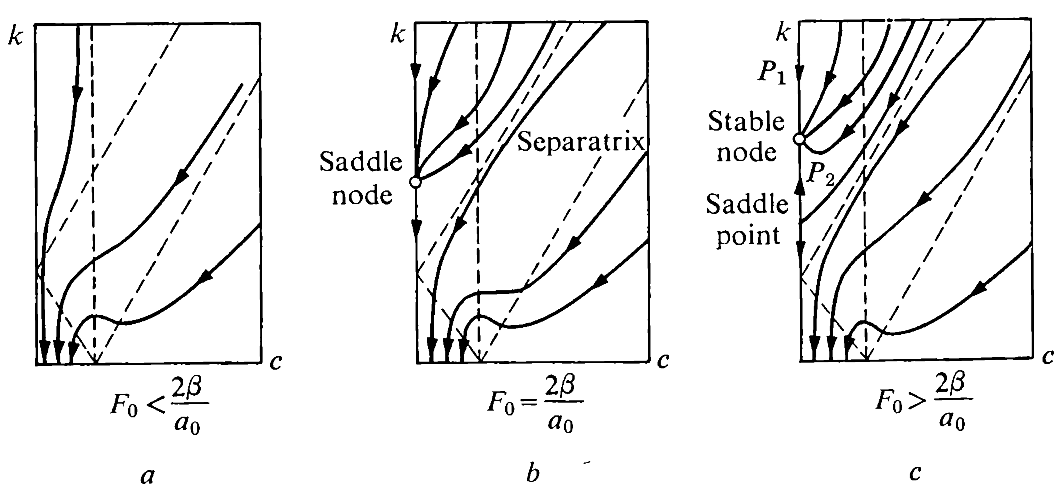

FIGURE 108

solution (for small h and A) and those obtained using the method of slowly varying parameters, we see that the expressions for the amplitude of the periodic solution agree with one another. It is clear that the expression* for T_0 obtained by the method of slowly changing coefficients is equal to $\cos T_0 = (2\beta^2/k_1^2) - 1$. By taking into account (219) we have that $T_0 = \beta\pi/2A_0$. Analogously, we find that $\sin \vartheta_0 = \beta\pi/2A_0$.† These expressions also agree with the expressions for T_0 and ϑ_0, which we found from the symmetric solution.

*The periodic motion satisfies the equation $\theta = a \cos \tau + b \sin \tau$, or $\theta = k_1 \cos \tau_1$, where $k_1 = \sqrt{a^2 + b^2}$, $\tau_1 = \tau - \varepsilon$, $\tan \varepsilon = b/a$.

When $\tau_1 = T_0/2$, we have $\theta = \beta$, consequently, $\beta = k_1 \cos T_0/2$, and hence, $\cos T_0/2 = \beta/k_1$ and $\cos T_0 = (2\beta^2/k_1^2) - 1$.

†The period $T = 2\,(T_0 + \vartheta_0)$, but $T = 2\pi$ and $T_0 + \vartheta_0 = \pi$.

IV

The Effect of an Exterior Harmonic Force on a Self-Excited System with One Degree of Freedom

The problem of the effect of an arbitrary external perturbation acting on a nonlinear system is one of the most complicated problems in the theory of nonlinear oscillations. The most conclusive results have been found for the case when the external force is a harmonic function of time. The study of the effect of an external force on nonlinear systems is given sufficiently broad coverage in the published works [3], [7], [18], [23], [25], [32], [35], [42].

In this chapter we shall examine only the problem of an external harmonic force on a self-excited system with one degree of freedom. This study is linked with one of the most important and interesting properties of self-excited systems, the phenomenon of synchronization, which is sometimes called "entrainment."

This phenomenon is related to the fact that for a sufficiently small difference in the frequency of the self-excited system and the frequency of the external force, the stable periodic motion of the system will assume the frequency of the external force. The external force imposes its rhythm on the self-excited system. The fundamental question, which the theory must answer, is the question of the magnitude ("the width") of the so-called interval of "entrainment," that is, the magnitude of the largest difference in the frequencies for which there will still be entrainment such that a further increase in the difference between the frequencies will result in the destruction of synchronization and the appearance of a new mode of behavior where we have a quasi-periodic motion with two basic frequencies in the system, one of which is the frequency of the external force

and the other being more or less the frequency of the self-excited system. (This is the so-called "beat" phenomenon.)

13. The Theory Effect of a Harmonic Force on an Oscillator with Rocard Collisions

By way of an example of a self-excited system which is subject to a harmonic external force, we choose an oscillator which is disturbed by impacts which strike the system in the direction of its motion each time it passes through its equilibrium position (60).

The equation of motion of this system which takes into account the disturbing (harmonic) force can be written in the form

$$m\ddot{x} + \gamma\dot{x} + cx = +A' \sin (pt + \psi),$$

where A', p, ψ are, respectively, the amplitude, the frequency, and the phase of the external force, γ is the coefficient of viscosity, m, c are constant parameters of the oscillator, and $I_0 = mv_0$ is the amplitude of the impulse from the impact of the external force.

We introduce the notation

$$2h = \frac{\gamma}{m}, \quad \omega^2 = \frac{c}{m}, \quad A = \frac{A'}{m},$$

and write the equation of motion in the form

$$\ddot{x} + 2h\dot{x} + \omega^2 x = A \sin (pt + \psi). \tag{220}$$

We choose the initial moment of time so that when $t = 0$, we have $x = 0$, $\dot{x} > 0$, and we suppose that at this moment the oscillator is affected by the given impulse. Then the velocity of the oscillator increases instantaneously to the magnitude v_0.

The general solution of Equation (220) has the form

$$\bar{x} = Be^{-ht} \sin (t\sqrt{\omega^2 - h^2} + \varphi) + \frac{A \sin (pt + \psi_0)}{\sqrt{(\omega^2 - p^2)^2 + 4h^2p^2}}, \tag{221}$$

where

$$\psi_0 = \psi - \varepsilon, \quad \tan \varepsilon = \frac{2hp}{\omega^2 - p^2}.$$

We impose another condition on the arbitrary quantity ψ. We shall require that when $t = 0$, we also have $\bar{x} = 0$. If this is true, the constants B, φ and ψ must satisfy the relation

$$B \sin \varphi + \frac{A \sin \psi_0}{\sqrt{(\omega^2 - p^2)^2 + 4p^2h^2}}. \tag{222}$$

We suppose that an impulse mv_0 acts upon the oscillator, which is moving according to the solution (221), where B, φ, and ψ_0 are related by (222), in such a way that at the moment $t = 0$ the velocity jumps to a quantity of magnitude v_0.

After the impact it is clear that the oscillator achieves a motion in accordance with the solution

$$x = Be^{-ht} \sin (t \sqrt{\omega^2 - h^2} + \varphi) + \frac{v_0}{\sqrt{\omega^2 - h^2}} e^{-ht} \sin t \sqrt{\omega^2 - h^2}$$
$$+ \frac{A \sin (pt + \psi_0)}{\sqrt{(\omega^2 - p^2)^2 + 4h^2p^2}}.* \tag{223}$$

This solution can be rewritten in the form

$$x = B_1e^{-ht} \sin [t \sqrt{\omega^2 - h^2} + \varphi - \varphi_0] + \frac{A \sin (pt + \psi_0)}{\sqrt{(\omega^2 - p^2)^2 + 4h^2p^2}}, \tag{224}$$

where

$$B_1 = \sqrt{\left(B + \frac{v_0 \cos \varphi}{\sqrt{\omega^2 - h^2}}\right)^2 + \left(\frac{v_0 \sin \varphi}{\sqrt{\omega^2 - h^2}}\right)^2}, \tag{225}$$

$$\tan \varphi_0 = \frac{v_0 \sin \varphi}{B \sqrt{\omega^2 - h^2} + v_0 \cos \varphi}. \tag{226}$$

The quantity φ_0 can have values ranging from $-\infty$ to ∞, in its dependence on the values of B and φ. Therefore, one can assume that φ_0 varies between $3/2\pi$ and $\pi/2$.

We are interested in the possibility of oscillations with the frequency of the external force. Periodic motion of the oscillator with the frequency p of the external force takes place if

$$-[x]^{t=\frac{\pi}{p}} = [\bar{x}]^{t=0}.*$$

Since $[\bar{x}]_{t=0} = 0$, by taking Equation (222) into consideration we get

$$B_1e^{-h\frac{\pi}{p}} \sin \left[\frac{\pi}{p} \sqrt{\omega^2 - h^2} + \varphi - \varphi_0\right] = -B \sin \varphi.$$

*It is easy to find this result; in defining the arbitrary constants of integration in the solution of (221), we take the following initial conditions: for $t = 0$, $x = 0$, and $x = \dot{x}_0 + v_0$, where $\dot{x}_0$ is the velocity of the oscillator at the moment $t = \theta$ before impact.

This will be partially satisfied if we put

$$\frac{\pi}{p}\sqrt{\omega^2 - h^2} + \varphi - \varphi_0 = \varphi + \pi, \tag{227}$$

$$B_1 e^{-h\frac{\pi}{p}} = B. \tag{228}$$

From (227) it follows that

$$\varphi_0 = \left(\frac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi, \tag{229}$$

and, hence, since φ_0 varies between $\pi/2$ and $3/2\pi$, we get

$$\frac{1}{2} \leqslant \frac{\sqrt{\omega^2 - h^2}}{p} - 1 \leqslant \frac{3}{2}.$$

By taking (229) into consideration, we rewrite (226) in the form

$$\frac{v_0 \sin\varphi}{\sqrt{\omega^2 - h^2}\left(B + \dfrac{v_0\cos\varphi}{\sqrt{\omega^2 - h^2}}\right)} = \tan\left[\left(\frac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right] \tag{230}$$

From the expressions (225) and (228) we get

$$\left(B + \frac{v_0\cos\varphi}{\sqrt{\omega^2 - h^2}}\right)^2 + \left(\frac{v_0\sin\varphi}{\sqrt{\omega^2 - h^2}}\right)^2 = B^2 e^{2h\frac{\pi}{p}} \tag{231}$$

From the equality (230) and (231) we find that

$$B = \frac{v_0}{\sqrt{\omega^2 - h^2}}\,\frac{1}{\sqrt{e^{2h\frac{\pi}{p}} - 2e^{h\frac{\pi}{p}}\cos\left[\left(\dfrac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right] + 1}},$$

$$\sin\varphi = \frac{e^{h\frac{\pi}{p}}\sin\left[\left(\dfrac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right]}{\sqrt{e^{2h\frac{\pi}{p}} - 2e^{h\frac{\pi}{p}}\cos\left[\left(\dfrac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right] + 1}}.$$

Substituting these expressions into (222) we get

* We are assuming that the impact acts upon the oscillator in the direction of its motion after half periods π/p each time the oscillator passes through its equilibrium position.

$$\frac{v_0}{\sqrt{\omega^2 - h^2}} \frac{e^{h\frac{\pi}{p}} \sin\left[\left(\frac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right]}{\left\{e^{2h\frac{\pi}{p}} - 2e^{h\frac{\pi}{p}} \cos\left[\left(\frac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right] + 1\right.}$$
$$= -\frac{A \sin \psi_0}{\sqrt{(\omega^2 - p^2)^2 + 4h^2p^2}}. \qquad (232)$$

The resulting relation contains all the results which are associated with the synchronization of the oscillator.

If the frequency p of the external force coincides with the "characteristic" frequency $\sqrt{\omega^2 - h^2}$ of the oscillator, then the left side of the expression (232) is equal to zero and synchronization takes place only for infinitely small values of A.

If we are concerned with finite values of A, then (232) is satisfied when $\sin \psi_0 = 0$. This means that the self-excited oscillations will coincide with the phase of the external force.

If we assume that A is fixed and increase the difference between p and $\sqrt{\omega^2 - h^2}$, then the moment will come when $\sin \psi_0 = \pm 1$. This determines the boundary of the region of "entrainment."

In order to get a relation between the quantity A (the amplitude of the external force) and the difference $\sqrt{\omega^2 - h^2}/p - 1$ on the boundary of the region of "entrainment," one must first set $\sin \psi_0 = \pm 1$.

We shall now assume that the friction in the system is small and that the frequency of the external force differs little from the frequency of the oscillator, that is, that

$$\frac{h}{p} \ll 1, \quad \frac{|\Delta\omega|}{p} \ll 1,$$

where $\Delta\omega = \omega - p$.

We further suppose that h and $\Delta\omega$ are of the same order in a neighborhood of zero. In this case

$$\sqrt{\omega^2 - h^2} = \omega \sqrt{1 - \frac{h^2}{\omega^2}} \approx \omega \approx p;$$

$$\left(\frac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi \approx \frac{\Delta\omega}{p}\pi;$$

$$\sin \frac{\Delta\omega}{p}\pi \approx \frac{\Delta\omega}{p}\pi;$$

$$\cos\frac{\Delta\omega}{p}\pi = 1 - 2\sin^2\left(\frac{\Delta\omega}{2p}\right)\pi \approx 1 - 2\left(\frac{\Delta\omega}{2p}\right)^2\pi^2;$$

$$e^{2\pi\frac{h}{p}} - 2e^{\pi\frac{h}{p}}\cos\left[\left(\frac{\sqrt{\omega^2 - h^2}}{p} - 1\right)\pi\right] + 1$$

$$\approx 1 + 2\pi\frac{h}{p} + \frac{1}{2}\frac{4\pi^2h^2}{p^2} - 2\left\{\left(1 + \frac{\pi h}{p} + \frac{1}{2}\frac{\pi^2h^2}{p^2}\right)\left[1 - 2\left(\frac{\Delta\omega}{2p}\right)^2\pi^2\right]\right\} + 1$$

$$\approx \frac{\pi^2}{p^2}[h^2 + (\Delta\omega)^2];$$

$$\sqrt{(\omega^2 - p^2)^2 + 4h^2p^2} \approx 2p\sqrt{(\Delta\omega)^2 + h^2}$$

Then formula (232) takes the form

$$\frac{2v_0 p}{\pi\sqrt{1 + \left(\frac{h}{\Delta\omega}\right)^2}} = -A\sin\psi_0.$$

Hence, it follows that the relation between the amplitude of the external force and $\Delta\omega$ on the border of the region of entrainment will be given by the formula

$$\frac{2v_0 p}{\pi\sqrt{1 + \left(\frac{h}{\Delta\omega}\right)^2}} = A, \tag{233}$$

and, hence,

$$\frac{\Delta\omega}{p} = \frac{A\pi h}{p\sqrt{4v_0^2p^2 - A^2\pi^2}}. \tag{234}$$

The theory of Rocard developed in this section does not explain the structure of the state space for the given system and does not answer the question of stability of the symmetric motion which is sudied. It is therefore of interest to look at this problem in more detail by using the method of slowly varying parameters.

14. The Method of Slowly Varying Parameters for Nonautonomous, Nonlinear Systems with One Degree of Freedom

Let the behavior of a dynamic system be described by the following differential equation

$$\ddot{x} + x = \mu f(x,\dot{x},t), \tag{235}$$

where μ is a small parameter which characterizes how closely the given system approximates a linear conservative one, $f(x,\dot{x},t)$ is a nonlinear function satisfying the property $f(x,\dot{x},t + 2\pi) = f(x,\dot{x},t)$, that is, $f(x,\dot{x},t)$ is a periodic function in the variable t, with period 2π.

When $\mu = 0$, Equation (235) has the solution

$$x = a \cos t + b \sin t. \tag{236}$$

If $\mu \neq 0$, we look for a solution of Equation (235) in the same form as (236), but where a and b are considered functions of time and where we suppose, just as we did previously, that the first derivative of x with respect to time will have the same form that it does if a and b are constants.

In this manner, we get

$$\dot{x} = -a \sin t + b \cos t, \tag{237}$$

$$\dot{a} \cos t + \dot{b} \sin t = 0. \tag{238}$$

The second derivative of x with respect to time is

$$\ddot{x} = -a \cos t - b \sin t - \dot{a} \sin t + \dot{b} \cos \mathrm{t}.$$

Substituting these expressions for $x, \dot{x}$, and $\ddot{x}$ into Equation (235), we have

$$-\dot{a} \sin t + \dot{b} \cos t = \mu f^*, \tag{239}$$

where

$$f^* = f(a \cos t + b \sin t, \quad -a \sin t + b \cos t, t).$$

Now we can find expressions for $\dot{a}$ and $\dot{b}$ by solving Equations (238) and (239) simultaneously:

$$\left.\begin{aligned} \dot{a} &= -\mu f^* \sin t, \\ \dot{b} &= \mu f^* \cos t. \end{aligned}\right\} \tag{240}$$

If we assume that a and b are slowly changing functions with respect to time, that is, that the variation of these functions during one period can be neglected, we can substitute the average values of these functions for one period in the right sides of the exact Equations (240), that is, instead of the system (240), we study the approximate system

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\mu}{2\pi} \int_0^{2\pi} f^* \sin t \, dt = P(a,b), \\ \frac{db}{dt} &= \frac{\mu}{2\pi} \int_0^{2\pi} f^* \cos t \, dt = Q(a,b). \end{aligned}\right\} \tag{241}$$

The system of Equations (241) is an autonomous system and can be studied by the method developed in Chapter 1. In the special case when

$$f(x,\dot{x},t) = \varphi(x,\dot{x}) + A_0 \sin t,$$

Equations (241) take the form

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\mu A_0}{2} - \frac{\mu}{2\pi}\int_0^{2\pi} \varphi^* \sin t\, dt, \\ \frac{db}{dt} &= \frac{\mu}{2\pi}\int_0^{2\pi} \varphi^* \cos t\, dt, \end{aligned}\right\} \tag{242}$$

where

$$\varphi^* = \varphi(a\cos t + b\sin t, \quad -a\sin t + b\cos t).$$

15. The Solution of the Problem of External Harmonic Impacts on an Oscillator by the Method of Slowly Varying Parameters

By way of a concrete example we shall study the case of synchronized small oscillations for an ordinary pendulum.

One can assume that this case is roughly analogous to the problem of synchronized clocks (with an autonomous discharge mechanism), which is powered by a sinusoidal electric current.

The equation for small oscillations of a pendulum has the form

$$ml^2\ddot{\varphi} + mgl\varphi + \gamma'\dot{\varphi} = I_0\dot{\varphi}\delta(\varphi) + A'\sin pt.$$

It is clear that the problem in its dynamic form is equivalent to the one studied in Section 13.

Here, m is the mass of the pendulum, l is its length, γ' is the coefficient of viscosity, $\delta(\varphi)$ is the Dirac delta function (Section 9), I_0 is the magnitude of the impulse of the impact, and p is the frequency of the external force.

Introducing the notations $\tau = pt$, $\omega^2 = g/l$ and $\bar{\omega}^2 = \omega^2/p^2$, we can rewrite the equation of motion in the form

$$\frac{d^2\varphi}{d\tau^2} + \bar{\omega}^2\varphi = \frac{A'}{ml^2p^2}\sin\tau - \frac{\gamma'}{ml^2p}\frac{d\varphi}{d\tau} + \frac{I_0}{ml^2p}\frac{d\varphi}{d\tau}\delta(\varphi).$$

We shall study the system when it is near "resonance," that is, we assume that

$$|\omega^2 - p^2| \ll 1.$$

We suppose that the dimensionless quantities satisfy the inequalities

$$\frac{A'}{ml^2p^2} \ll 1; \quad \frac{\gamma'}{ml^2p} \ll 1; \quad \frac{I_0}{ml^2p} \ll 1.$$

We introduce into our study the "disturbance factor" ζ by using the equation

$$\bar{\omega}^2 = 1 + \mu\zeta,$$

Then the equation of motion takes the form

$$\frac{d^2\varphi}{d\tau^2} + \varphi = \mu \left[A_0 \sin\tau - \frac{d\varphi}{d\tau} + I\frac{d\varphi}{d\tau}\,\delta(\varphi) - \zeta\varphi\right], \tag{243}$$

where

$$\mu = \frac{\gamma'}{ml^2p}, \quad A_0 = \frac{A'}{\gamma' p}, \quad I = \frac{I_0}{\gamma'}.$$

If we are looking for a solution of Equation (243) in the form (236), that is, in the form

$$\varphi = a\cos\tau + b\sin\tau, \quad \frac{d\varphi}{d\tau} = -a\sin\tau + b\cos\tau, \tag{244}$$

the Equations (242) become

$$\frac{da}{d\tau} = \mu\left[-\frac{A_0}{2} - \frac{a}{2} + \frac{\zeta b}{2} - \frac{I}{2\pi}\int_{-0}^{+0} \sin\xi'\,\delta(\varphi)\,d\varphi - \right.$$

$$\left. - \frac{I}{2\pi}\int_{+0}^{-0} \sin\xi''\,\delta(\varphi)\,d\varphi\right],$$

$$\frac{db}{d\tau} = \mu\left[-\frac{b}{2} - \frac{\zeta a}{2} + \frac{I}{2\pi}\int_{-0}^{+0} \cos\xi'\delta(\varphi)\,d\varphi + \right.$$

$$\left. + \frac{I}{2\pi}\int_{+0}^{-0} \cos\xi''\delta(\varphi)\,d\varphi\right],$$

where ξ' and ξ'' are the roots of the equation

$$a\cos\xi + b\sin\xi = 0.$$

If we use the notations

$$\frac{2I}{\pi} = \alpha, \quad \frac{a}{\alpha} = x, \quad \frac{b}{\alpha} = y, \quad \frac{A_0}{\alpha} = A, \quad \tau_1 = \frac{2}{\mu}\tau,$$

then, finally, we get

$$\left.\begin{aligned}\frac{dx}{d\tau_1} &= -A - x + \zeta y + \frac{x}{\sqrt{x^2+y^2}} = P(x,y),\\ \frac{dy}{d\tau_1} &= -y - \zeta x + \frac{y}{\sqrt{x^2+y^2}} = Q(x,y).\end{aligned}\right\} \tag{245}$$

From the expression (xy) one can see that a singular point on the xy-plane will correspond to a limit cycle in the original system. Limit cycles on the xy-plane correspond to quasi-periodic motions (the "beat" phenomenon) of the pendulum.

Singular points of the system (245) are determined by the equations

$$-x_0\left(1 - \frac{1}{\sqrt{x_0^2+y_0^2}}\right) + \zeta y_0 = A,$$

$$-y_0\left(1 - \frac{1}{\sqrt{x_0^2+y_0^2}}\right) - \zeta x_0 = 0.$$

By squaring both of these equations and combining them we get

$$\rho_0^2\left(1 - \frac{1}{\rho_0}\right)^2 + \zeta^2\rho_0^2 = A^2$$

or

$$\rho_0^2 - 2\rho_0 + 1 = A^2 - \zeta\rho_0^2, \tag{246}$$

where

$$\rho_0^2 = x_0^2 + y_0^2.$$

Expression (246) is called the equation of the "resonance" curve. The values of ρ_0, which are roots of this equation correspond to periodic motions in the original system with the frequency of the external force.

The coordinates of the singular points are given by ρ_0 in the following way:

$$x_0 = \frac{\rho_0(1-\rho_0)}{A}, \quad y_0 = \frac{\zeta\rho_0^2}{A}.$$

Let us go on to determine the type of the singular point.

As is well known (Section 3), the type of a singular point is determined by the sign of the quantities

$$p = -\{P'_x(x_0,y_0) + Q'_y(x_0,y_0)\},$$

$$q = P'_x(x_0,y_0)\,Q'_y(x_0,y_0) - P'_y(x_0,y_0)\,Q'_x(x_0,y_0).$$

In this instance

$$P'_x = -1 + \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{\sqrt{(x^2+y^2)^3}},\; Q'_x = -\zeta - \frac{xy}{\sqrt{(x^2+y^2)^3}},$$

$$P'_y = \zeta - \frac{xy}{\sqrt{(x^2+y^2)^3}}, \quad Q'_y = -1 + \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{(x^2+y^2)^3}},$$

and, hence, we have

$$p = 2 - \frac{1}{\rho_0}, \quad q = 1 - \frac{1}{\rho_0} + \zeta^2, \quad \Delta = \frac{1}{\rho_0^2} - 4\zeta^2.$$

Figure 109 is a diagram of the type of singular points on the $\zeta\rho_0$-plane. The region of saddle points is determined by the curve $q = 0$, that is

$$\rho_0 = \frac{1}{1+\zeta^2}$$

When $q > 0$, the stable singular points are separated from the unstable ones by the curve $p = 0$, that is, $\rho_0 = 1/2$. The boundary between the foci and the nodes is given by the equation $\Delta = 0$, that is,

$$\left(\frac{1}{\rho_0} - 2\zeta\right)\left(\frac{1}{\rho_0} + 2\zeta\right) = 0.$$

We go on to the construction of the resonance curves on the $\zeta\rho_0$-plane for various fixed values of A^2.

By solving Equation (246) for ρ_0, we get

$$\rho_0 = \frac{1 \pm \sqrt{A^2(1+\zeta)^2 - \zeta^2}}{1+\zeta^2}.$$

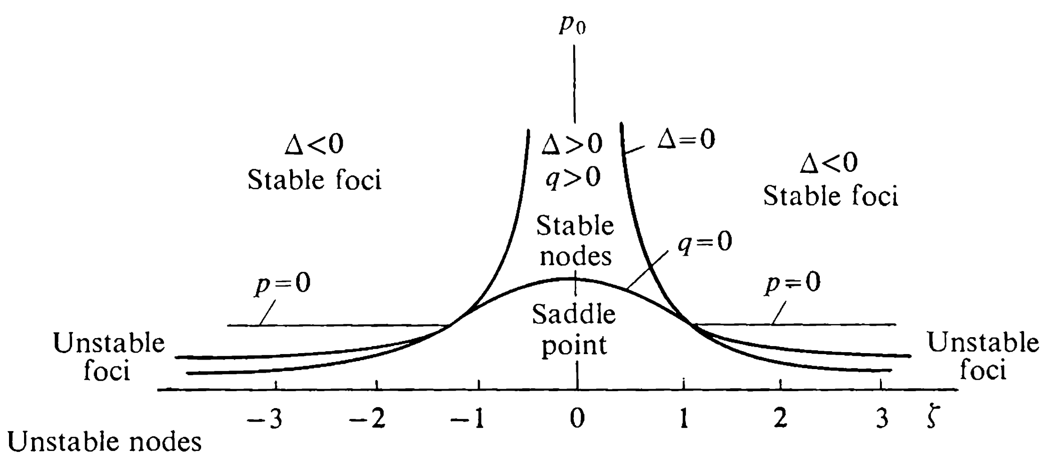

FIGURE 109

From this expression it is clear that when $A^2 > 1$ and for arbitrary ζ, there will be only one positive value of ρ_0. When $A^2 < 1$, for $\zeta^2 < A^2/1 - A^2$ we will have two positive roots, and for $\zeta^2 > A^2/1 - A^2$ there will be no real roots.

We differentiate expression (246) with respect to ρ_0, and obtain

$$\frac{d\zeta}{d\rho_0} = \frac{1 - \rho_0 - \rho_0\zeta^2}{\rho_0^2\zeta}.$$

Consequently, the locus of points for which the tangents of the resonance curves are equal is

$$\rho_0 = \frac{1}{1 + \zeta^2};$$

that is, it coincides with the curve $q = 0$.

We find the points of intersection of the resonance curve with the curve $q = 0$ by solving the system of equations

$$\rho_0^2 - 2\rho_0 + 1 - A^2 + \zeta^2\rho_0^2 = 0,$$

$$\rho_0 = \frac{1}{1 + \zeta^2},$$

for ζ. We get

$$\zeta^2 = \frac{A^2}{1 - A^2}.$$

This means that when $A^2 > 1$, the resonance curve does not intersect the curve $q = 0$, and, when $A^2 < 1$, there will be two points of intersection at

$$\zeta_1 = \frac{A}{\sqrt{1 - A^2}}, \quad \zeta_2 = -\frac{A}{\sqrt{1 - A^2}}.$$

Now, we note that the curves $p = 0$, $q = 0$, and $\Delta = 0$ intersect at the points $\zeta = \pm 1$, $\rho_0 = 1/2$. We find the points of intersection of the resonance curve with the curve $p = 0$ by solving the equations

$$\rho_0^2 - 2\rho_0 + 1 + \zeta^2\rho_0^2 - A^2 = 0,$$

$$\rho_0 = \frac{1}{2}.$$

The values of ζ for these points are as follows:

$$\zeta_1' = \sqrt{4A^2 - 1}, \quad \zeta_2' = -\sqrt{4A^2 - 1}.$$

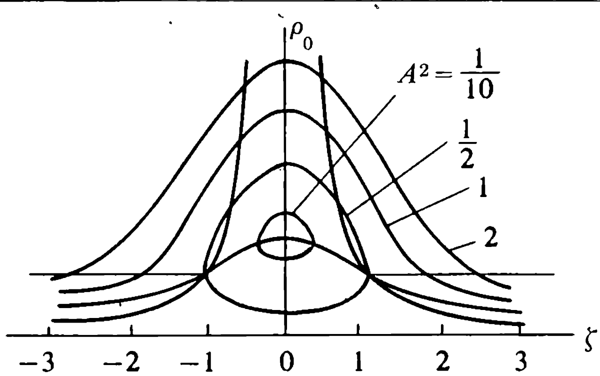

FIGURE 110

In Figure 110 the resonance curves for various values of A^2 are drawn on the $\zeta\rho_0$-plane

When $A^2 < ½$, the resonance curves have one stable portion which is above the curve $q = 0$, that is, ρ_0 for this portion of the resonance curve corresponds to stable singular points and consequently to stable periodic oscillations in the original system with the frequency of the external force.

When $A^2 > ½$, the resonance curves have one stable portion which is above the curve $p = 0$.

Let us go on to explain the mechanism of entrainment. We shall restrict ourselves to the case of weak signals, that is, $A^2 < ¼$.

In polar coordinates the system (245) has the form:

$$\left.\begin{aligned} \frac{d\rho}{d\tau_1} &= \frac{1}{\rho}[-Ax - \rho^2 + \rho], \\ \frac{d\varphi}{d\tau_1} &= \frac{1}{\rho^2}[-\zeta\rho^2 + Ay], \end{aligned}\right\} \tag{247}$$

where

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi.$$

When ρ is sufficiently large we have $d\rho/d\tau < 0$, which means that infinity is unstable.

The singular points on the xy-plane are at the intersections of the curves

$$-Ax - \rho^2 + \rho = 0,$$

$$-\zeta\rho^2 + Ay = 0.$$

The first curve does not depend on ζ since we have

$$-Ax - (x^2 + y^2) + \sqrt{x^2 + y^2} = 0.$$

In polar coordinates this curve has the equation

$$\rho(-A\cos\varphi - \rho + 1) = 0. \tag{248}$$

The equation of the second curve is

$$Ay - \zeta(x^2 + y^2) = 0$$

and can be rewritten in the form

$$x^2 + \left(y - \frac{A}{2\zeta}\right)^2 = \left(\frac{A}{2\zeta}\right)^2, \tag{249}$$

that is, the second curve is a circle with its center at the point $x = 0, y = A/2\zeta$. In Figure 111 the curves (248) and (249) are constructed for $A^2 = 1/16$ and for various values of ζ.

Let us examine Figure 111 and Figure 112. When $\zeta = 0$, we have three singular points: the first is a stable node, the second is a saddle point, and the type of the third (at $x = 0$, $y = 0$) singular point is as yet undetermined.

When $0 < \zeta < \zeta_1$, again there are three points of a similar type. When $\zeta = \zeta_1$, the stable node and the saddle point merge. The point $x = 0$, $y = 0$ remains the same. When $\zeta > \zeta_1$, there is only one singular point at $x = 0$, $y = 0$.

In order to determine the type of this point we write the first equation of the system (247) in the form

$$\frac{d\rho}{d\tau_1} = -A\cos\varphi - \rho + 1.$$

When $A < 1$, one can choose $\rho = \rho_1$ so that it will satisfy $\rho_1 < 1 - A \cos\varphi$ and that for all $\rho < \rho_1$ we will have

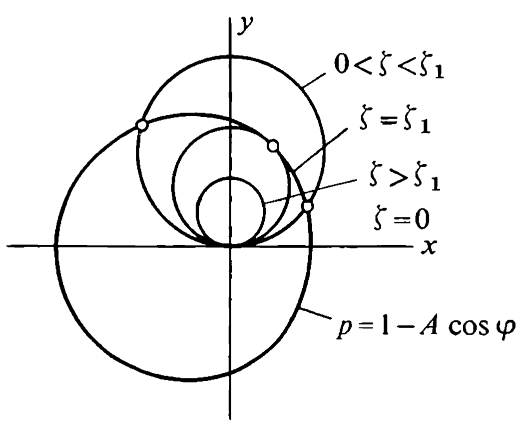

FIGURE 111

$$\frac{d\rho}{d\tau_1} > 0.$$

Thus the circles $\rho = c$ where $c < \rho_1$ about the point $x = 0, y = 0$ form a system of cycles without any tangents, that is, a system of closed curves, at each point of which the velocity vector in the state plane is directed either always inwards or always outwards. Consequently, all the trajectories move away from the point $x = 0$, $y = 0$ with increasing τ_1. Since this is true when $A < 1$ and for arbitrary ζ, one can say that the type of the point at $x = 0, y = 0$ does not change when the two other points, the node and the saddle point, coalesce.

To show the existence and uniqueness of the limit cycle on the xy-plane and to establish the boundaries of the limit cycle, we shall make use of the curve-of-contacts method and a theorem of Poincaré and Dulac. We shall explain briefly the idea of the curve-of-contacts method [2].

Let there be given a family of closed, nonintersecting curves which fill up the plane,

$$F(x,y) = c.$$

We shall call this system of curves a *topographic system*. A curve of contaets is a curve which is the locus of points of tangency of the curves of the topographic system with the integral curves of the dynamic system. Assume that we can choose the topographic system in such a way that the curve of contacts is closed. Then we draw the largest and smallest curves of the topographic system which are tangent to the curve of contacts. It is obvious that all the curves of the topographic system which lie outside the largest curve and inside the smallest curve will be cycles without tangents. (This is not entirely obvious; the reader should make certain that he understands the situation.)

Therefore if limit cycles exist, then they will be located in the ring-

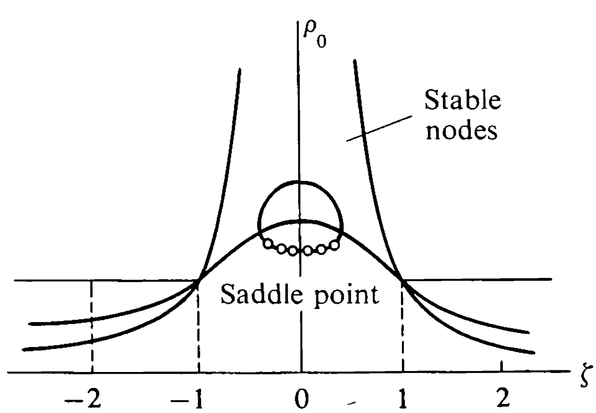

FIGURE 112

shaped region which is bounded by the two selected curves of the topographic system. A sufficient condition for the existence of at least one cycle is the condition that state velocity vector on the selected curves of the topographic system be directed either always inwards or always outwards with respect to the corresponding ring-shaped region.

Let the dynamic system be described by the equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y).$$

Then the equation of the curve of contacts takes the form

$$\frac{P}{Q} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial x}},$$

since

$$\frac{\partial F}{\partial x}\frac{dx}{dy} + \frac{\partial F}{\partial y} = 0.$$

If the topographic system is a family of circles $x^2 + y^2 = c$, then the equation for the curve of contacts is

$$\frac{P}{Q} = -\frac{y}{x}.$$

For our problem we choose as a topographic system the family of circles $x^2 + y^2 = R^2$. Then the equation for the curve of contacts becomes

$$-\frac{x}{y} = \frac{-y - \zeta x + \dfrac{y}{\sqrt{x^2 + y^2}}}{-A - x + \zeta y + \dfrac{x}{\sqrt{x^2 + y^2}}}$$

or

$$Ax + (x^2 + y^2) - \sqrt{x^2 + y^2} = 0.$$

In polar coordinates this equation takes the form

$$\rho = 1 - A\cos\varphi.$$

The radii of the extremal circles of the topographic system which are tangent to the curve of contacts are

$$R_1 = 1 - A,$$

$$R_2 = 1 + A.$$

Consequently, if limit cycles exist, then they lie inside the ring-shaped region described by the circles with radii R_1 and R_2.

By applying the theorem of Poincaré and Dulac [55] we shall prove that there is one limit cycle. This theorem can be formulated in the following manner. Let there be given a system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y),$$

where the functions $P(x,y)$ and $Q(x,y)$ are assumed to be single-valued and to have derivatives, and let $F(x,y)$ be some single-valued differentiable function. Then, if

$$B(x,y) = \frac{\partial}{\partial x}(PF) + \frac{\partial}{\partial y}(QF)$$

does not change sign in some ring-shaped region D, there will be at most one limit cycle inside D.

In our example the function

$$B(x,y) = \frac{\partial}{\partial x}(PF) + \frac{\partial}{\partial y}(QF) = \frac{1}{\rho} - 2,$$

where we choose $F = 1$, does not change sign inside the ring between the extremal curves of the topographic system whenever $A < ½$ since in this case the curve $1/\rho - 2 = 0$ lies inside the smaller circle of the topographic system (Figure 113). Consequently, in the given ring-shaped region there is at most one limit cycle.

A qualitative picture of the xy-plane for various values of ζ is shown in Figure 114.

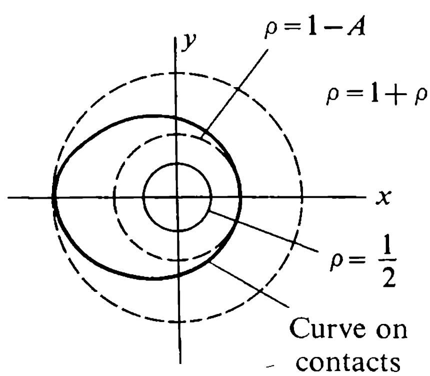

FIGURE 113

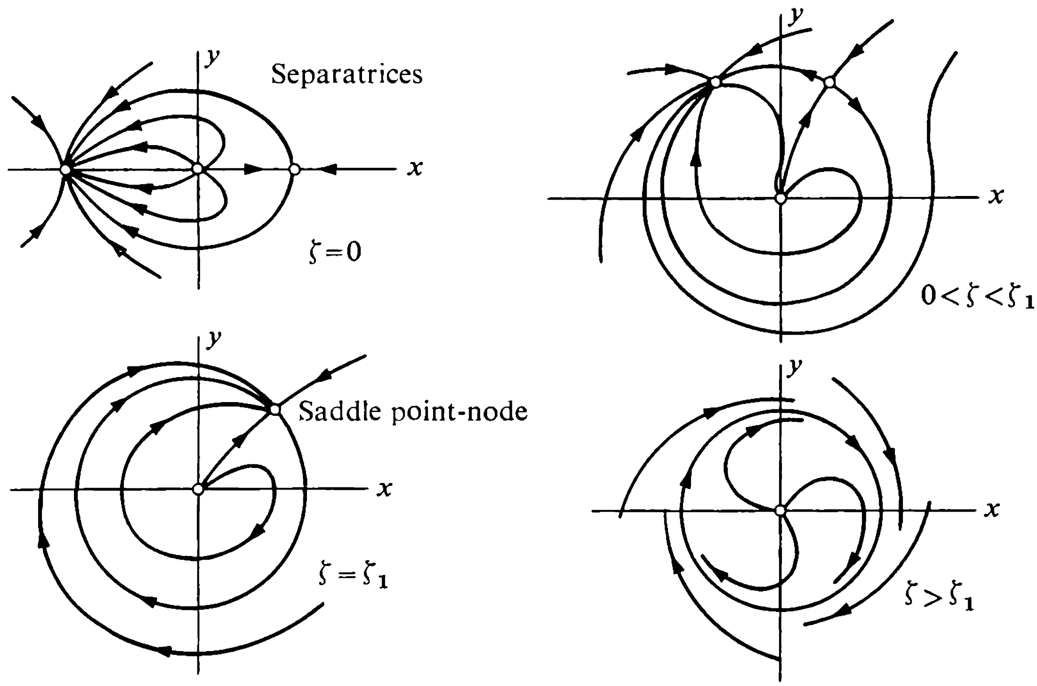

FIGURE 114

We shall determine the relation between the amplitude of the external force and $\Delta\omega = \omega - p$ on the boundary of the region of entrainment.

From the relation $\bar{\omega}^2 = 1 + \mu\zeta$ we get

$$\frac{(\omega - p)(\omega + p)}{p^2} = \mu\zeta.$$

On the boundary of the region of entrainment

$$\zeta = \frac{A}{\sqrt{1 - A^2}}.$$

Since $A = A_0/\alpha$, and $A_0 = A'/\gamma' p$, $\alpha = 2I/\pi$, $I = I_0/\gamma'$, $\mu = \gamma'/ml^2p$, we have that

$$2\Delta\omega = \frac{\gamma' A' \pi}{ml^2\sqrt{4I_0^2p^2 - \pi^2 A'^2}}.$$

Using the notation of Section 13, that is,

$$\frac{A'}{ml^2} = A, \quad \frac{\gamma'}{ml^2} = 2h, \quad ml^2\nu_0 = I_0,$$

we get

$$A = \frac{2v_0 p}{\pi\sqrt{1 + \left(\frac{h}{\Delta\omega}\right)^2}} \quad \text{and} \quad \frac{\Delta\omega}{p} = \frac{Ah\pi}{p\sqrt{4v_0^2 p^2 - A^2\pi^2}}.$$

The latter expression is in complete agreement with the expression of (234).

For sufficiently small A one can assume that it is approximately true that $\zeta = A$. In this case we have that

$$\frac{\Delta\omega}{p} = \frac{1}{2mgl}\frac{A'}{R},$$

where $R = 2I/\pi$ is the amplitude of the self-excited oscillations.

16. The Effect of Delay on Forced Synchronization

Let us study small oscillations of a pendulum under the condition that there is an external sinusoidal force acting on it.

For small oscillations the equation of motion has the form (Section 15)

$$ml^2\ddot{\varphi} + mql\varphi = A' \sin pt - \gamma'\dot{\varphi} + E(\dot{\varphi}).$$

We shall assume that the function $E(\dot{\varphi})$, which describes the disturbing force, is delayed. This means that the argument $\Delta(t)$ of this function is to be replaced by $\dot{\varphi}(t - \Delta t)$.

If there were no delay, then the function could be approximated in the following way:

$$E(\dot{\varphi}) = \begin{cases} E' \text{ when } \dot{\varphi} > 0; \\ -E' \text{ when } \dot{\varphi} < 0. \end{cases}$$

We recall the notations used in Section 15 and we assume that the given system closely approximates a linear conservative one. Then we obtain an equation for the motion of the pendulum which has the form (243)

$$\frac{d^2\varphi}{d\tau^2} + \varphi = \mu\left[A_0 \sin\tau - \frac{d\varphi}{d\tau} - \zeta\varphi + E^*\left(\frac{d\varphi}{d\tau}\right)\right], \tag{250}$$

where $\mu = \gamma'/ml^2p$ is a small parameter which characterizes how closely the given system approximates a linear conservative one and $E^* = E/\gamma'p$.

If we look for a solution of Equation (250) in the form (236), that is, the form

$$\varphi = a\cos\tau + b\sin\tau, \tag{251}$$

then Equation (242) for this case takes the form

$$\frac{da}{d\tau} = \frac{\mu}{2}\left(-a + \zeta b - A_0 + \frac{4E_0}{\pi}\frac{a\cos\Delta - b\sin\Delta}{\sqrt{a^2+b^2}}\right),$$

$$\frac{db}{d\tau} = \frac{\mu}{2}\left(-b - \zeta a + \frac{4E_0}{\pi}\frac{a\sin\Delta + b\cos\Delta}{\sqrt{a^2+b^2}}\right),$$

where $\Delta = p\Delta t$, $E_0 = E'/\gamma' p$.

Introducing the notations

$$B = \frac{4E_0}{\pi}, \quad x = \frac{a}{B}, \quad y = \frac{b}{B}, \quad A = \frac{A_0}{B}, \quad \tau_1 = \tau\frac{2}{\mu},$$

we have

$$\left.\begin{aligned} \frac{dx}{d\tau_1} &= -x + \zeta y - A + \frac{x\cos\Delta - y\sin\Delta}{\sqrt{x^2+y^2}} = P(x,y); \\ \frac{dy}{d\tau_1} &= -y - \zeta x + \frac{x\sin\Delta + y\cos\Delta}{\sqrt{x^2+y^2}} = Q(x,y). \end{aligned}\right\} \tag{252}$$

We recall that according to (251) the stable equilibrium states (the singular points) of the system (252) on the xy-plane correspond to periodic motions in the original system, with the frequency of the applied force (the state of entrainment). The stable limit cycles on the xy-plane correspond to "beats" in the original system.

By making the change of variable

$$x = \rho\cos\theta, \quad y = \rho\sin\theta,$$

we can rewrite Equation (252) in the form

$$\left.\begin{aligned} \frac{d\rho}{d\tau_1} &= -\rho + \cos\Delta - A\cos\theta, \\ \rho\frac{d\theta}{d\tau_1} &= -\zeta\rho + \sin\Delta + A\sin\theta. \end{aligned}\right\} \tag{253}$$

We find the singular points of the system from the equations

$$-\rho + \cos\Delta - A\cos\theta = 0,$$

$$-\zeta\rho + \sin\Delta + A\sin\theta = 0.$$

We eliminate θ from these equations and obtain an equation for the resonance curve

$$\rho^2(1+\zeta^2) - 2\rho(\cos\Delta + \zeta\sin\Delta) + 1 = A^2. \tag{254}$$

The values of ρ which satisfy this equation correspond to the singular points of the system of Equations (253).

The characteristic equation for the equations of the first approximation of the system (253) has the form

$$s^2 + \left(2 - \frac{\cos \Delta}{\rho}\right)s + 1 + \zeta^2 - \frac{1}{\rho}(\cos \Delta + \zeta \sin \Delta) = 0.$$

Here ρ has the value which corresponds to the singular point which we are studying.

The signs of the quantities

$$p = 2 - \frac{\cos \Delta}{\rho}, \quad q = 1 + \zeta^2 - \frac{1}{\rho}(\cos \Delta + \zeta \sin \Delta),$$

$$\delta = p^2 - 4q = \frac{1}{\rho^2} - \left(2\zeta - \frac{\sin \Delta}{\rho}\right)^2$$

determine the type of the equilibrium states. When $q < 0$, the equilibrium states are unstable (saddle points). When $q > 0$ $p > 0$, the equilibrium states are stable, and when $q > 0$ $p < 0$ the equilibrium states are unstable. When $\delta > 0$, the equilibrium states are nodes, and when $\delta < 0$ they are foci.

Let us examine the $\zeta\rho$-plane. On this plane the curve $q = 0$ determines the region of saddle points. When $q > 0$ the curve $p = 0$ divides the stable equilibrium states from the unstable equilibrium states. The boundary between the foci and the nodes is given by the equation $\delta = 0$, that is, by

$$\left[\frac{1}{\rho} - \left(2\zeta - \frac{\sin \Delta}{\rho}\right)\right]\left[\frac{1}{\rho} + \left(2\zeta - \frac{\sin \Delta}{\rho}\right)\right] = 0. \tag{255}$$

The curve $p = 0$ is a straight line

$$\rho = \tfrac{1}{2} \cos \Delta. \tag{256}$$

The equation for the curve $q = 0$ has the form

$$\rho = \frac{\cos \Delta + \zeta \sin \Delta}{1 + \zeta^2}. \tag{257}$$

For the curve $q = 0$ we have that

$$\rho = 0 \quad \text{when} \quad \zeta = -\cot \Delta,$$

$$\max \rho = \cos^2 \frac{\Delta}{2} \quad \text{when} \quad \zeta = \tan \frac{\Delta}{2},$$

and

$$\rho \to 0 \quad \text{when} \quad \zeta \to \infty.$$

The curve $q = 0$ intersects the straight line

$$\rho = \tfrac{1}{2} \cos \Delta \quad (p = 0)$$

for the following values of ζ:

$$\zeta = \frac{1 + \sin \Delta}{\cos \Delta}, \quad \zeta = -\frac{1 - \sin \Delta}{\cos \Delta}.$$

Let us go on to a study of the resonance curve. Solving Equation (254) for ρ, we get

$$\rho_{1,2} = \frac{\cos \Delta + \zeta \sin \Delta \pm \sqrt{A^2(1 + \zeta^2) - (\sin \Delta - \zeta \cos \Delta)^2}}{1 + \zeta^2}. \tag{258}$$

When $A = 0$, the resonance curve contracts to the point with coordinates

$$\zeta = \tan \Delta, \quad \rho = \cos \Delta.$$

From expression (258) it follows that when $A > 1$, ρ will have one positive value, and when $A < 1$ it will have two positive values. From (254) we have that

$$\frac{d\rho}{d\zeta} = \frac{\rho(\sin \Delta - \zeta\rho)}{\rho(1 + \zeta^2) - (\cos \Delta + \zeta \sin \Delta)},$$

that is, the resonance curve intersects the curve $q = 0$ where it has a vertical tangent.

The intersection of the resonance curve and the straight line $p = 0$ occurs for the following values of ζ:

$$\zeta_{1,2} = \frac{2 \sin \Delta \pm \sqrt{4A^2 - \cos^2 \Delta}}{\cos \Delta}, \tag{259}$$

and, hence, it follows that the resonance curve does not intersect the line $p = 0$ when $A < (1/2) \cos \Delta$.

Solving Equations (254) and (257) simultaneously we find the values of ζ for the points of intersection of the resonance curve and the curve $q = 0$,

$$\zeta_{3,4} = \frac{\cos \Delta \sin \Delta \pm \sqrt{A^2(1 - A^2)}}{\cos^2 \Delta - A^2}, \tag{260}$$

that is, the resonance curve intersects the curve $q = 0$ only when $A^2 < 1$.

The resonance curve with A_1^2 determined by the expression

$$A_1^2 = \tfrac{1}{2}(1 - \sin \Delta),$$

passes through the point of intersection of the curve $q = 0$ and the line

$p = 0$ with coordinates

$$\rho = \frac{\cos \Delta}{2}, \quad \zeta = \frac{1 + \sin \Delta}{\cos \Delta}$$

The resonance curve with

$$A_2^2 = \tfrac{1}{2}(1 + \sin \Delta),$$

passes through the point

$$\rho = \frac{\cos \Delta}{2}, \quad \zeta = -\frac{1 - \sin \Delta}{\cos \Delta}.$$

Figure 115 shows the resonance curves on the $\zeta\rho$-plane for various fixed values of A and with $\Delta = 0.5$. From the construction it follows that when $A^2 > A_2^2$, the resonance curve has one stable portion which lies above the curve $q = 0$ and the line $p = 0$. Consequently, the system performs a stable harmonic oscillation with the frequency of the external force for those ζ which satisfy the inequality

$$\frac{2 \sin \Delta - \sqrt{4A^2 - \cos^2 \Delta}}{\cos \Delta} < \zeta < \frac{2 \sin \Delta + \sqrt{4A^2 - \cos^2 \Delta}}{\cos \Delta},$$

that is, for those values of ζ for which there is entrainment. When $A_1^2 < A^2 < A_2^2$, the phenomenon of entrainment will occur when

$$\frac{\cos \Delta \sin \Delta - \sqrt{A^2(1 - A^2)}}{\cos^2 \Delta - A^2} < \zeta < \frac{2 \sin \Delta + \sqrt{4A^2 - \cos^2 \Delta}}{\cos \Delta}.$$

Finally, when $0 < A^2 < A_1^2$, there will be entrainment if

$$\frac{\cos \Delta \sin \Delta - \sqrt{A^2(1 - A^2)}}{\cos^2 \Delta - A^2} < \zeta < \frac{\cos \Delta \sin \Delta + \sqrt{A^2(1 - A^2)}}{\cos^2 \Delta - A^2}.$$

We shall clarify the mechanism of entrainment. To do this we study the behavior of the integral curves on the xy-plane. From (253) we conclude that for sufficiently large ρ, we have that

$$\frac{d\rho}{d\tau_1} < 0,$$

that is, infinity is unstable.

We see from Equations (252) that the singular points on the xy-plane are at the points of intersection of the curves

$$P(x,y) = 0, \quad Q(x,y) = 0$$

or, from Equations (253),

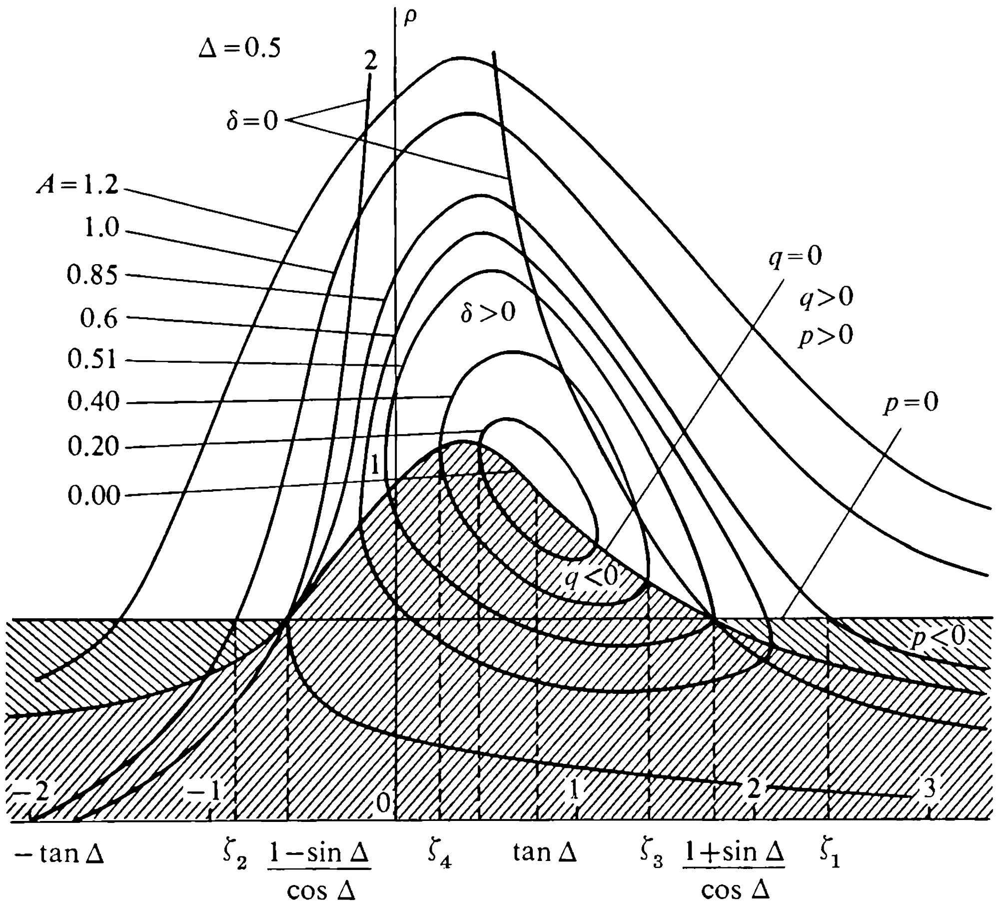

FIGURE 115

$$\rho = \cos\Delta - A\cos\theta, \tag{261}$$

$$\rho = \frac{\sin\Delta}{\zeta} + \frac{A}{\zeta}\sin\theta. \tag{262}$$

We shall study Figures 116 and 117. Figure 116 shows the curves (261) and (262) when $A = 0.2$ and for various values of ζ.

When $\zeta = \tan\Delta$, there are three singular points: one is a stable node, the second is a saddle point, and the type of the third ($x = 0, y = 0$) is not as yet determined.

When $\tan\Delta < \zeta < \zeta_3$, $\zeta_4 < \zeta < \tan\Delta$, again there are three singular points.

When $\zeta = \zeta_3$, $\zeta = \zeta_4$ the stable node and the saddle point merge into one equilibrium state. The point $x = 0, y = 0$ remains the same.

When $\zeta > \zeta_3$, $\zeta < \zeta_4$, there is only one singular point at $x = 0, y = 0$.

To clarify the character of the singular point at $x = 0, y = 0$ we look at the first equation of the system (253),

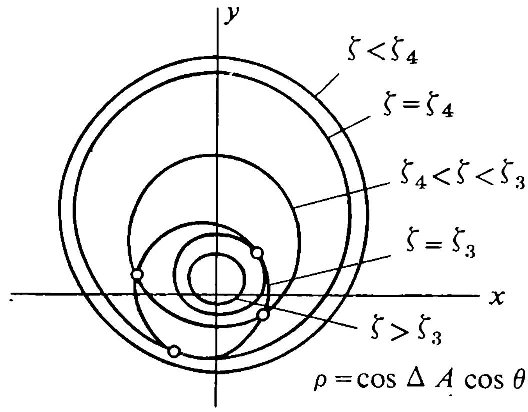

FIGURE 116

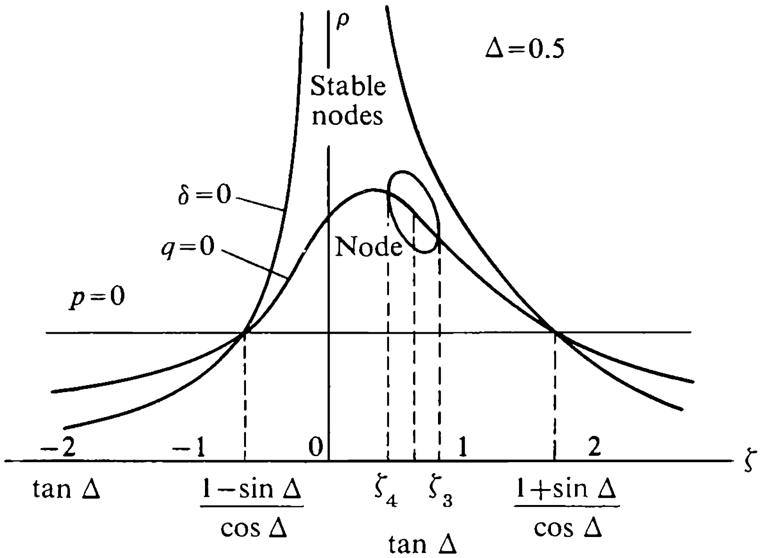

FIGURE 117

$$\frac{d\rho}{d\tau_1} = -\rho + \cos\Delta - A\cos\theta.$$

If $A < \cos\Delta$, then one can choose a ρ_1 which satisfies the inequality

$$\rho_1 < \cos\Delta - A\cos\theta$$

and, hence, for all $\rho < \rho_1$ we have that

$$\frac{d\rho}{d\tau_1} > 0.$$

Consequently, the circles $\rho = c$ where $c < \rho_1$ about the point $x = 0$, $y = 0$ form a system of cycles without contacts and all trajectories move away from the point $x = 0, y = 0$ as the variable τ_1 increases.

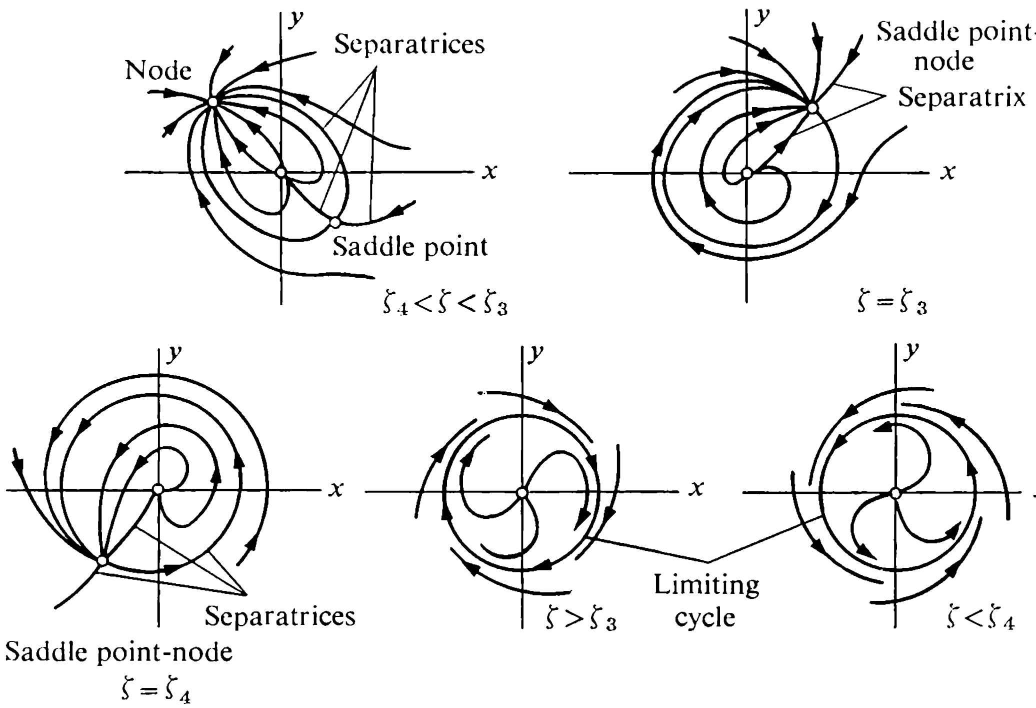

FIGURE 118

Since this is true for all ζ when $A < \cos \Delta$, it follows that the type of the point $x = 0, y = 0$ for these values of A does not change when the other points, the node and the saddle point, coalesce.

Now, we shall prove that "beats" may exist in this mechanical system. As a topographic system we choose the family of circle

$$x^2 + y^2 = R^2.$$

The curve of contacts will have the form

$$-\frac{x}{y} = \frac{-y - \zeta x + (x \sin \Delta + y \cos \Delta)/\sqrt{x^2 + y^2}}{-x + \zeta y - A + (x \cos \Delta - y \sin \Delta)/\sqrt{x^2 + y^2}}$$

or

$$x^2 + y^2 + Ax - \sqrt{x^2 + y^2} \cos \Delta = 0.$$

In polar coordinates this relation can be reduced to the following form:

$$\rho = \cos \Delta - A \cos \theta.$$

The radii of the extremal circles of the topographic system, which are tangent to the curve of contacts, are therefore

$$R_1 = \cos \Delta - A; \quad R_2 = \cos \Delta + A.$$

The function

$$B(x,y) = \frac{\partial}{\partial x}(pF) + \frac{\partial}{\partial y}(QF) = \frac{1}{\rho}\cos\Delta - 2$$

(where we choose $F = 1$) does not change sign inside the annulus between the circles of the topographic system for $A < (½) \cos \Delta$ because the curve $(\cos \Delta/\rho) - 2 = 0$ in this case lies inside the smaller circle of the topographic system.

When $A < (½) \cos \Delta$ there will be at most one stable limit cycle in the annulus between the extremal circles of the topographic system when $\zeta > \zeta_3$ and $\zeta < \zeta_4$. And, when

$$\frac{\cos\Delta}{2} < A < \sqrt{\frac{1-\sin\Delta}{2}}$$

again there will be a limit cycle since all the trajectories move away from the point $x = 0, y = 0$; infinity is unstable and, when $\zeta > \zeta_3$ and $\zeta < \zeta_4$, there are no singular points except $x = 0, y = 0$.

For those Δ which satisfy the inequality

$$0 < \arcsin\Delta < \tfrac{1}{2},$$

with

$$\sqrt{\frac{1-\sin\Delta}{2}} < A < \sqrt{\frac{1+\sin\Delta}{2}}$$

there will likewise be a limit cycle if $\zeta < \zeta_4$. Figure 118 gives a qualitative picture of the xy-plane for various values of ζ when $A < (½) \cos \Delta$.

V

The Effect of an Exterior Harmonic Force on a Self-Excited System with Two Degrees of Freedom

The study of the behavior of a self-excited system with several degrees of freedom [4], [15], [16], [24], [35], and [52] under the influence of an external sinusoidal force is a very complicated mathematical problem. For this reason we shall restrict ourselves to a study of a nonlinear system with two degrees of freedom which is closely approximated by a linear conservative system. Furthermore, we shall study only two cases: first, the case where the normal frequencies of the linear system are much greater than the frequency of the excitation; second, the case where one of the natural frequencies of the linear system is equal to the frequency of the applied sinusoidal force.

17. The Method of Slowly Varying Parameters for Nonautonomous Systems with Two Degrees of Freedom

First, we shall study the case when the natural (normal) frequencies of the linear system are much greater than the frequency of the external force.

Let the equations for the dynamic system have the form

$$\left.\begin{aligned} \ddot{x} + A_1\ddot{y} + B_1 y + n_1^2 x &= \mu f(x,\dot{x},y,\dot{y}) + D_1 \sin t + E_1 \cos t, \\ \ddot{y} + A_2\ddot{x} + B_2 x + n_2^2 y &= \mu g(x,\dot{x},y,\dot{y}) + D_2 \sin t + E_2 \cos t, \end{aligned}\right\} \tag{263}$$

where $A_1, A_2, B_1, B_2, n_1^2, n_2^2, D_1, D_2, E_1, E_2$ are constant quantities, where f and g are nonlinear functions, and where μ is a small parameter which

characterizes how closely the given system approximates a linear conservative one.

If $\mu = 0$, the solution of the system (263) has the form

$$\left.\begin{aligned} x &= a\sin(k_1t+\beta_1) + b\sin(k_2t+\beta_2) + d_1\sin t + e_1\cos t,\\ y &= \alpha_1 a\sin(k_1t+\beta_1) + \alpha_2 b\sin(k_2t+\beta_2) + d_2\sin t + e_2\cos t,\end{aligned}\right\} \quad (264)$$

where a,b and β_1,β_2 are arbitrary constants of integration, k_1 and k_2 are the natural frequencies of the linear homogeneous system (determined by Equation (131)), and where α_1 and α_2 are "distribution" coefficients (determined by the expressions (132)).

$$d_1 = \frac{D_1(n_2^2-1) - D_2(B_1-A_1)}{\Delta};$$

$$e_1 = \frac{E_1(n_2^2-1) - E_2(B_1-A_1)}{\Delta};$$

$$d_2 = \frac{D_2(n_1^2-1) - D_1(B_2-A_2)}{\Delta};$$

$$e_2 = \frac{E_2(n_1^2-1) - E_1(B_2-A_2)}{\Delta};$$

$$\Delta = \sigma - (n_1^2 + n_2^2 - A_1B_2 - A_2B_1) + n_1^2n_2^2 - B_1B_2,$$
$$\sigma = 1 - A_1A_2.$$

In a manner analogous to the technique used in Section 9 we look for a solution to the system (263) when $\mu \neq 0$ in the form (264) assuming that a, b, β_1, and β_2 are functions of time.

Proceeding in the same way as we did in Section 9 we get equations of the same form as (140) through (143). Hence, the functions f^* and g^* have the form

$$\left.\begin{aligned} f^* = f(&a\sin\xi + b\sin\eta + d_1\sin t + e_1\cos t,\\ &ak_1\cos\xi + bk_2\cos\eta + d_1\cos t - e_1\sin t,\\ &\alpha_1 a\sin\xi + \alpha_2 b\sin\eta + d_2\sin t + e_2\cos t,\\ &\alpha_1 ak_1\cos\xi + \alpha_2 bk_2\cos\eta + d_2\cos t - e_2\sin t);\\ g^* = g(&a\sin\xi + b\sin\eta + d_1\sin t + e_1\cos t,\\ &ak_1\cos\xi + bk_2\cos\eta + d_1\cos t - e_1\sin t,\\ &\alpha_1 a\sin\xi + \alpha_2 b\sin\eta + d_2\sin t + e_2\cos t,\\ &\alpha_1 ak_1\cos\xi + \alpha_2 bk_2\cos\eta + d_2\cos t - e_2\sin t);\end{aligned}\right\}$$

where

$$\xi = k_1 t + \beta_1, \quad \eta = k_2 t + \beta_2.$$

If we assume that a, b, β_1, and β_2 are slowly varying functions with respect to time, we can take the average of the right sides of the resulting equations over the periods $2\pi/k_1$, $2\pi/k_2$, and 2π.

The approximate equations for determining a, b, β_1, and β_2 will have the form of Equations (144) and (145) where

$$F_1 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\xi \, d\xi \, d\eta \, dt, \qquad G_1 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \cos\xi \, d\xi \, d\eta \, dt;$$

$$F_2 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\eta \, d\xi \, d\eta \, dt, \qquad G_2 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \cos\eta \, d\xi \, d\eta \, dt;$$

$$F_3 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\xi \, d\xi \, d\eta \, dt, \qquad G_3 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\xi \, d\xi \, d\eta \, dt;$$

$$F_4 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\eta \, d\xi \, d\eta \, dt, \qquad G_4 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\eta \, d\xi \, d\eta \, dt.$$

Let us now look at the second case when one of the basic frequencies of the linear system is equal to the frequency of the external force. We suppose that the amplitude of the external force has the same order in a neighborhood of zero as μ.

Let the equations for the motion be

$$\left.\begin{aligned} \ddot{x} + A_1\ddot{y} + B_1 y + n_1^2 x &= \mu f(x,\dot{x},y,\dot{y},t), \\ \ddot{y} + A_2\ddot{x} + B_2 x + n_2^2 y &= \mu g(x,\dot{x},y,\dot{y},t), \end{aligned}\right\} \tag{265}$$

where the nonlinear functions f and g have the property that

$$f(x,\dot{x},y,\dot{y},t + 2\pi) = f(x,\dot{x},y,\dot{y},t);$$

$$g(x,\dot{x},y,\dot{y},t + 2\pi) = f(x,\dot{x},y,\dot{y},t).$$

Suppose that $k_2^2 > k_1^2 = 1$; we look for the solution of the system (265) in the form

$$x = a\sin(t + \varphi) + b\sin(k_2 t + \psi),$$

$$y = \alpha_1 a\sin(t + \varphi) + \alpha_2 b\sin(k_2 t + \psi)$$

or, in the more convenient form,

$$\left.\begin{aligned} x &= a_1 \sin t + a_2 \cos t + b \sin (k_2 + \psi), \\ y &= \alpha_1 a_1 \sin t + \alpha_1 a_2 \cos t + \alpha_2 b \sin (k_2 t + \psi), \end{aligned}\right\} \tag{266}$$

where α_1, α_2, b, and ψ are assumed to be slowly varying functions of time.

In the same way as before we find the following approximate equations for α_1, α_2, b, and ψ:

$$\left.\begin{aligned} \frac{da_1}{dt} &= \frac{\mu}{2\sigma (k_2^2 - 1)} \left[\frac{A_2 - B_2}{\alpha_1} F_1^* + (A_1 - B_1) G_1^*\right], \\ \frac{da_2}{dt} &= -\frac{\mu}{2\sigma (k_2^2 - 1)} \left[\frac{A_2 - B_2}{\alpha_1} F_2^* + (A_1 - B_1) G_2^*\right], \\ \frac{db}{dt} &= -\frac{\mu}{2k_2\sigma (k_2^2 - 1)} \left[\frac{A_2 k_2^2 - B_2}{\alpha_2} F_3^* + (A_1 k_2^2 - B_1) G_3^*\right], \\ b\frac{d\psi}{dt} &= \frac{\mu}{2k_2\sigma (k_2^2 - 1)} \left[\frac{A_2 k_2^2 - B_2}{\alpha_2} F_4^* + (A_1 k_2^2 - B_1) G_4^*\right], \end{aligned}\right\} \tag{267}$$

$$F_1^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \cos t \, d\xi \, dt, \qquad G_1^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \cos t \, d\xi \, dt;$$

$$F_2^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \sin t \, d\xi \, dt, \qquad G_2^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \sin t \, d\xi \, dt;$$

$$F_3^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \cos \xi \, d\xi \, dt, \qquad G_3^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \cos \xi \, d\xi \, dt;$$

$$F_4^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \sin \xi \, d\xi \, dt, \qquad G_4^* = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \sin \xi \, d\xi \, dt;$$

$$\begin{aligned} f^* = f\,(&a_1 \sin t + a_2 \cos t + b \sin \xi, \\ &a_1 \cos t - a_2 \sin t + k_2 b \cos \xi, \\ &\alpha_1 a_1 \sin t + \alpha_1 a_2 \cos t + \alpha_2 b \sin \xi, \\ &\alpha_1 a_1 \cos t - \alpha_1 a_2 \sin t + \alpha_2 k_2 b \cos \xi, t); \end{aligned}$$

$$\begin{aligned} g^* = g\,(&a_1 \sin t + a_2 \cos t + b \sin \xi, \\ &a_1 \cos t - a_2 \sin t + k_2 b \cos \xi, \end{aligned}$$

$$\alpha_1 a_1 \sin t + \alpha_1 a_2 \cos t + \alpha_2 b \sin \xi,$$
$$\alpha_1 a_1 \cos t - \alpha_1 a_2 \sin t + \alpha_2 k_2 b \cos \xi, t);$$
$$\xi = k_2 t + \psi.$$

18. Forced Oscillations in a Complex Generator

We shall study the theory of forced oscillations in a vacuum-tube generator. Neglecting the grid current one can write Kirchoff's equations for the two circuits shown in Figure 119 in the form

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + r_1 i_1 + \frac{1}{c_1}\int_0^t i_1 dt &= M\frac{di_a}{dt} + N\frac{di_2}{dt} + P_0 \sin pt; \\ L_2 \frac{di_2}{dt} + r_2 i_2 + \frac{1}{c_2}\int_0^t i_2 dt &= N\frac{di_1}{dt}. \end{aligned}\right\} \quad (268)^*$$

We use the notation

$$v_1 = \frac{1}{c_1}\int_0^t i_1 dt, \quad v_2 = \frac{1}{c_2}\int_0^t i_2 dt$$

and assume that the tube is characterized by a cubic parabola†

$$i_a = S v_1 \left(1 - \frac{v_1^2}{3K^2}\right),$$

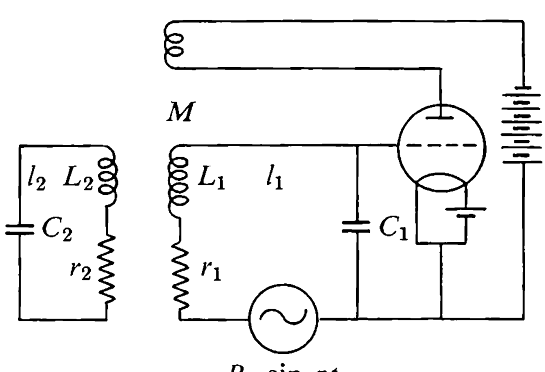

FIGURE 119

* The notations are shown in Figure 119.

† It is assumed that the current across the plate depends only on the grid voltage.

where S is the so-called steepness of the tube characteristic and where K is the saturation voltage. Then, we can rewrite Equations (268) in the form

$$\left.\begin{aligned} \ddot{v}_1 + n_1^2 v_1 - Nc_2 n_1^2 \ddot{v}_2 &= P_0 n_1^2 \sin pt + n_1^2 \left\{ MS - r_1 c_1 - MS \frac{v_1^2}{K^2} \right\} \dot{v}_1; \\ \ddot{v}_2 + n_2^2 v_2 - Nc_1 n_2^2 \ddot{v}_1 &= -r_2 c_2 n_2^2 \dot{v}_2, \end{aligned}\right\} \quad (269)$$

where

$$n_1^2 = \frac{1}{L_1 c_1}, \quad n_2^2 = \frac{1}{L_2 c_2}.$$

We introduce the dimensionless time $\tau = pt$, the dimensionless variables

$$x = \frac{v_1}{K}\sqrt{\frac{MS}{MS - r_1 c_1}}, \qquad y = \frac{v_2}{K}\sqrt{\frac{MS}{MS - r_1 c_1}},$$

and the dimensionless parameters

$$\mu = (MS - r_1 c_1)\frac{n_1^2}{p}, \quad r = \frac{r_2}{L_2 (MS - r_1 c_1) n_1^2},$$

$$\lambda = \frac{P_0 n_1^2}{Kp_2^2}\sqrt{\frac{MS}{MS - r_1 c_1}}, \quad \kappa_1 = Nc_2 n_1^2, \quad \kappa_2 = Nc_1 n_2^2.$$

Then we can rewrite Equations (269) in the form

$$\left.\begin{aligned} \frac{d^2x}{d\tau^2} - \kappa_1 \frac{d^2y}{d\tau^2} + \bar{n}_1^2 x &= \lambda \sin \tau + \mu (1 - x^2)\frac{dx}{d\tau}, \\ \frac{d^2y}{d\tau^2} - \kappa_2 \frac{d^2x}{d\tau^2} + \bar{n}_2^2 y &= -\mu r \frac{dy}{d\tau}, \end{aligned}\right\} \quad (270)$$

where

$$\bar{n}_1^2 = \frac{n_1^2}{p^2}, \quad \bar{n}_2^2 = \frac{n_2^2}{p^2}.$$

We assume that μ is a small parameter which characterizes how closely the given system approximates a linear one.*

If one is looking for a solution of the system (27) in the form (264), then the Equations (144) will have the form

* We shall study only the first quadrant of the uv-plane because $u = a^2 > 0$, $v = b^2 > 0$.

$$\left.\begin{aligned}\frac{du}{d\tau} &= Au\,(a_0^2 - u - 2v - 2d_1^2),\\ \frac{dv}{d\tau} &= Bv\,(b_0^2 - 2u - v - 2d_1^2),\end{aligned}\right\} \tag{271}$$

where

$$u = a^2, \quad v = b^2, \quad A = \frac{1}{4}\frac{\mu}{\sigma}\frac{n_2^2 - k_1^2}{k_2^2 - k_1^2}, \quad B = \frac{1}{4}\frac{\mu}{\sigma}\frac{k_2^2 - n_2^2}{k_2^2 - k_1^2},$$

$$a_0^2 = 4 - 4r\frac{n_1^2 - k_1^2}{n_2^2 - k_1^2}, \quad b_0^2 = 4 - 4r\frac{k_2^2 - n_1^2}{k_2^2 - n_2^2},$$

$$d_1 = \frac{\lambda\,(\bar{n}_2^2 - 1)}{\sigma - (\bar{n}_1^2 + \bar{n}_2^2) + \bar{n}_1^2\bar{n}_2^2}.$$

From the equation for the natural frequencies of the linear system

$$\sigma k^4 - (n_1^2 + n_2^2)\,k^2 + n_1^2 n_2^2 = 0,$$

where $\sigma = 1 - \kappa_1\kappa_2$, it follows that

$$k_2^2 > n_1^2,\; k_2^2 > n_2^2 \quad \text{and} \quad k_1^2 < n_1^2,\; k_1^2 < n_2^2,$$

and, consequently, $A > 0$, $B > 0$.

From Equations (145) it follows that

$$\beta_1 = \text{const}, \;\; \beta_2 = \text{const}.$$

In order to answer the question about the possible forms of motion in the generator for various values of the parameters, one must study the possible subdivisions of the uv-plane into trajectories which, in turn, are determined by the location and type of the singular points, the limit cycles, and the separatrices of the Equations (271).

We introduce the notation

$$\alpha = a_0^2 - 2d_1^2,$$
$$\beta = b_0^2 - 2d_1^2.$$

The equations for the singular points have the form

$$u\,(\alpha - u - 2v) = 0,$$
$$v\,(\beta - 2u - v) = 0.$$

By solving these equations we obtain the singular points:

$$P_1\colon u_1 = 0,\ v_1 = 0;$$

$$P_2\colon u_2 = \alpha,\ v_2 = 0;$$

$$P_3\colon u_3 = 0,\ v_3 = \beta;$$

$$P_4\colon u_4 = \frac{1}{3}(2\beta - \alpha),\ v_4 = \frac{1}{3}(2\alpha - \beta).$$

The roots of the first approximation of the characteristic equation for the point P_1 are $s_1 = A\alpha$, $s_2 = B\beta$; for the point P_2 are $s_1 = -A\alpha$, $s_2 = B(\beta - 2\alpha)$; and for the point P_3 are $s_1 = A(\alpha - 2\beta)$, $s_2 = -B\beta$. For the point P_4, given the condition that $u_4 > 0$, $v_4 > 0$, the roots of the characteristic equation always have opposite signs and, hence, if the point P_4 exists, it will be a saddle point.

From (264) one can see that the point P_1 corresponds in the original system to motion with the frequency of the external force; the point P_2 corresponds to motion with two frequencies, p and k_1; the point P_3 corresponds to motion with two frequencies p and k_2; and the point P_4 corresponds to motion with three frequencies, p, k_1, and k_2. Since the singular point P_4 is unstable, motion with three frequencies, p_1, k_1, and k_2 is always unstable.

The type of the singular points for various values of the parameters of the system and of d_1^2 are shown in Table 3.

From an examination of this table it is easy to find the bifurcated values of the parameters and the character of the variation of steady-state motions in the generator with increasing values of the amplitudes of the external force.

In case 1 the generator passes from a state of rest for $d_1 = 0$ to a periodic motion with the frequency of the external force for $d_1 > 0$.

In cases 2 and 3, when there is an absence of an external force $d_1 = 0$, the generator has a periodic motion with the frequency k_1. Under the influence of an external force there will be biharmonic motion with the frequencies k_1 and p, but when $d_1^2 > a_0^2/2$, there will be periodic motion with the frequency p.

In case 4 when $d_1 = 0$ (in the absence of an external force) the system can have either one of two periodic motions, either with a frequency of k_1 or k_2, depending on the initial conditions. When the amplitude of the external force is small (when $d_1^2 < b_0^2 - a_0^2/2$) the system can have either one of two biharmonic motions either with the frequencies p and k_1 or with the frequencies p and k_2. If d_1^2 is further increased, only one biharmonic motion is possible with the frequencies k_1 and p (when $b_0^2/2 > d_1^2 > b_0^2 - a_0^2/2$). And if the amplitude of the external force is

Table 3

Case	$d_1 = 0$	P_1	P_2	P_3	P_4
1. $a_0^2 < 0$, $b_0^2 < 0$	$d_1 = 0$	stable node	none	none	none
	$d_1^2 > 0$	stable node	none	none	none
2. $a_0^2 > 0$, $b_0^2 < 0$	$d_1 = 0$	saddle point	stable node	none	none
	$\frac{a_0^2}{2} > d_1^2 > 0$	saddle point	stable node	none	none
	$d_1^2 > \frac{a_0^2}{2}$	stable node	none	none	none
3. $a_0^2 > 2b_0^2$, $b_0^2 > 0$	$d_1 = 0$	unstable node	stable node	saddle point	none
	$\frac{b_0^2}{2} > d_1^2 > 0$	unstable node	stable node	saddle point	none
	$\frac{a_0^2}{2} > d_1^2 > \frac{b_0^2}{2}$	saddle point	stable node	none	none
	$d_1^2 > \frac{a_0^2}{2}$	stable node	none	none	none
4. $a_0^2 > b_0^2$, $b_0^2 > 0$, $2b_0^2 > a_0^2$	$d_1 = 0$	unstable node	stable node	stable node	saddle point
	$\frac{2b_0^2 - a_0^2}{2} > d_1^2 > 0$	unstable node	stable node	stable node	saddle point
	$\frac{b_0^2}{2} > d_1^2 > \frac{2b_0^2 - a_0^2}{2}$	unstable node	stable node	saddle point	none
	$\frac{a_0^2}{2} > d_1^2 > \frac{b_0^2}{2}$	saddle point	stable node	none	none
	$d_1^2 > \frac{a_0^2}{2}$	stable node	none	none	none

The cases when $b_0^2 > a_0^2, 2a_0^2 > b_0^2$; $b_0^2 > 2a_0^2, a_0^2 > 0$; $a_0^2 < 0, b_0^2 > 0$. are analogous

further increased ($d_1^2 > a_0^2/2$), there can be only one periodic motion with the frequency p (the frequency of the external force).

We will not stop to study the behavior of the generator with variation in the frequency and the amplitude of the external force. (For a more detailed study of this see reference [35].)

We shall study the case when $k_1 = p$.

We suppose that in the Equations (270)

$$\lambda = \mu\lambda_1,$$

where

$$\lambda_1 = \frac{P_0}{Kp(MS - r_1c_1)}\sqrt{\frac{MS}{MS - r_1c_1}}.$$

We shall look for a solution of the Equations (270) in the form (266), that is, in the form

$$\begin{aligned} x &= a_1 \sin\tau + a_2\cos\tau + b\sin(\bar{k}_2\tau + \psi), \\ y &= \alpha_1 a_1 \sin\tau + \alpha_2 a_2 \cos\tau + \alpha_2 b \sin(\bar{k}_2\tau + \psi), \end{aligned} \qquad (272)$$

where $\bar{k}_2 = k_2/p$ and $\bar{k}_1 = k_1/p = 1$ are the roots of the equation

$$\sigma\bar{k}^4 - (\bar{n}_1^2 + n_2^2)\,\bar{k}^2 + \bar{n}_1^2\bar{n}_2^2 = 0.$$

Equations (267) become

$$\left.\begin{aligned} \frac{du}{d\tau_1} &= u\,[a_0^2 - (u^2 + v^2 + 2w)] - 4\lambda_1, \\ \frac{dv}{d\tau_1} &= v\,[a_0^2 - (u^2 + v^2 + 2w)], \\ \frac{dw}{d\tau_1} &= Ew\,[b_0^2 - (2u^2 + 2v^2 + w)], \end{aligned}\right\} \qquad (273)$$

where

$$u = a_2, \quad v = a_1, \quad w = b^2 > 0,$$

$$\tau_1 = \frac{\mu}{8\sigma}\frac{p^2 - n_2^2}{p^2 - k_2^2}\,\tau, \quad E = \frac{1}{2}\frac{n_2^2 - k_2^2}{p^2 - n_2^2} > 0,$$

From (272) we have:

$$a_0^2 = 4 - 4r\frac{p^2 - n_1^2}{p^2 - n_2^2}, \quad b_0^2 = 4 - 4r\,\frac{n_1^2 - k_2^2}{n_2^2 - k_2^2}.$$

1) the singular points of the system (273) on the uv-plane correspond to periodic motions in the generator with the frequency $k_1 = p$;

2) singular points on the w-axis correspond to periodic motions with the frequency k_2;

3) singular points outside the uv-plane and outside the w-axis correspond to biharmonic motions with the frequencies $k_1 = p$ and k_2.

If we let $\rho = u^2 + v^2$, then the equations for the singular points have the form

$$u^2 = \rho, \quad v = 0, \tag{274}$$

$$\rho[a_0^2 - (\rho + 2w)]^2 = 16\lambda_1^2, \tag{275}$$

and

$$w(b_0^2 - 2\rho - w) = 0. \tag{276}$$

From these equations it follows that the singular points on the uv-plane lie on the u-axis and are determined by the equation

$$\rho[a_0^2 - \rho] = 16\lambda_1^2, \tag{277}$$

which, at the same time, determines the amplitude curve of the corresponding periodic motions with the frequency $k_1 = p : R^2 = \rho(\lambda_1)$.

A condition for the stability of singular points is obtained from the characteristic equation,

$$[E(b_0^2 - 2R^2) - s][s^2 - 2(a_0^2 - 2R^2)s + (a_0^2 - R^2)(a_0^2 - 3R^2)] = 0,$$

which has the roots

$$s_1 = E(b_0^2 - 2R^2), \quad s_2 = a_0^2 - R^2, \quad s_3 = a_0^2 - 3R^2. \tag{278}$$

Thus, the conditions for stability are the inequalities

$$R^2 > \frac{b_0^2}{2}, \quad R^2 > a_0^2. \tag{279}$$

From Equation (275) it follows that there are no singular points on the w-axis and, hence, if the generator is under the influence of an external force with the frequency $p = k_1$, periodic motion with the frequency k_2 is impossible.

Singular points which correspond to biharmonic motions with frequencies of p and k_2 are determined by Equations (275) and (276).

The square of the amplitude for these equations is equal to

$$R^2 = w + \rho = b_0^2 - \rho.$$

Consequently, the equation for the amplitude of the curve is

$$16\lambda_1^2 = (b_0 - R^2)[a_0^2 + b_0^2 - 3R^2]^2. \tag{280}$$

Because of the fact that

$$w = b_0^2 - 2\rho \quad \text{and} \quad R^2 = b_0^2 - \rho,$$

(280) makes sense only when

$$\frac{b_0^2}{2} < R^2 < b_0^2.$$

The characteristic equation for these motions has the form

$$\begin{aligned}(s - a_0^2 - b_0^2 + 3R^2)\{s^2 + s[E(2R^2 - b_0^2) + b_0^2 - a_0^2 + R^2] \\ + E(2R^2 - b_0^2)[9R^2 - (a_0^2 + 7b_0^2)]\} = 0.\end{aligned} \tag{281}$$

Since for real motions we have that $R^2 > b_0^2/2$, it follows from (281) that, when

$$R^2 < \frac{a_0^2 + 7b_0^2}{9}$$

the singular points are unstable and in the opposite case they are stable.

Let us look at the equilibrium states of the generator as the quantity λ_1 varies (the amplitude of the external influence).

We introduce the notations

$$16\lambda_{11}^2 = 4\left(\frac{a_0^2}{3}\right)^3, \; 16\lambda_{12}^2 = \tfrac{1}{8} b_0^2(2a_0^2 - b_0^2)^2,$$
$$16\lambda_{13}^2 = \tfrac{4}{81}(2b_0^2 - a_0^2)^3.$$

We shall look at several of the most typical cases.

1. $a_0^2 < 0$, $b_0^2 < 0$ or $a_0^2 > 0$, $b_0^2 > 0$. The generator will have periodic motion with the frequency $p = k_1$. There will be no biharmonic motions.

2. $a_0^2 < 0$, $b_0^2 > 0$. Here, there are two possibilities:

   a) $5b_0^2 + 2a_0^2 < 0$. When $\lambda_1 < \lambda_{12}$, there is biharmonic motion with the frequencies $p = k_1$ and k_2. When $\lambda_1 > \lambda_{12}$, there will be only harmonic motion with the frequency $p = k_1$.

   b) $5b_0^2 + 2a_0^2 > 0$. Stable biharmonic motion is possible when $\lambda_1 < \lambda_{13}$. Periodic motion with the frequency $p = k_1$ is possible when $\lambda_1 > \lambda_{12}$. Therefore, in this case there can be hysteresis with respect to λ_1 (in the interval $\lambda_{12} < \lambda < \lambda_{13}$). The amplitude curves are drawn in Figure 120. The stable portions of these curves are indicated by the thicker lines.

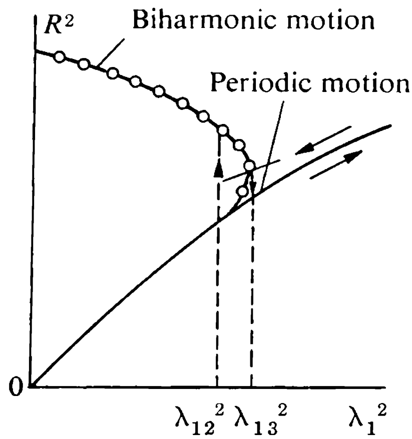

FIGURE 120

3. $0 < b_0^2 < a_0^2/2$. In relation to the stable motions this case coincides with the case when $a_0^2 > 0$, $b_0^2 < 0$.

4. $0 < a_0^2 < b_0^2/2$. The curve of the amplitudes for $\lambda_{12} < \lambda_1 < \lambda_{13}$ is shown in Figure 121.

5. $0 < a_0^2 < 2b_0^2$ or $0 < b_0^2 < 2a_0^2$. The most typical subcase here is shown in Figure 122. When $\lambda_1 > \lambda_{13}$, the generator can have either a periodic motion with the frequency $p = k_1$, or a biharmonic motion with the frequencies $p = k_1$ and k_2, depending on its initial state. The initial condition depends on the history of the generator. It could have had a periodic motion with the frequency of k_1 or with the frequency of k_2.

This case is interesting because of the possibility of shifting the generator from the frequency k_2 to k_1. To do this it is sufficient to make $\lambda_1 > \lambda_{13}$ and to remove the excitation.

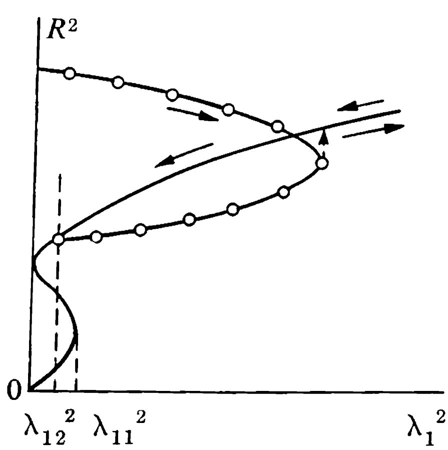

FIGURE 121

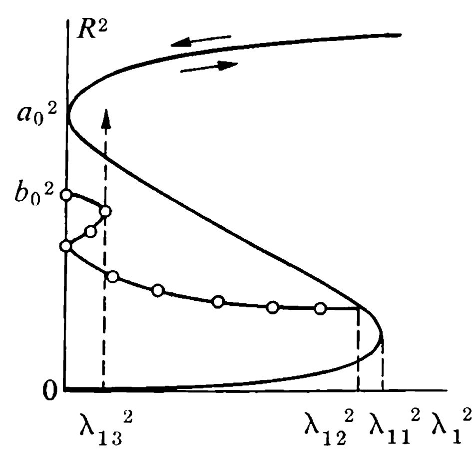

FIGURE 122

19. The Method of Slowly Varying Parameters for a System with Gyroscopic Forces

1. Let the natural frequencies of the linear system be far away from the frequency of the external influence.

Let the equations for the dynamic system have the form

$$\left.\begin{aligned} \ddot{x} - \lambda_1 \dot{y} \pm n_1^2 x &= \mu f(x,\dot{x},y,\dot{y}), \\ \ddot{y} + \lambda_2 \dot{x} \pm n_2^2 y &= \mu g(x,\dot{x},y,\dot{y}) + Q \sin t. \end{aligned}\right\} \tag{282}$$

If one is looking for a solution of this equation in the form

$$x = a \sin(k_1 t + \beta_1) + b \sin(k_2 t + \beta_2) + d \cos t,$$

$$y = \alpha_1 a \cos(k_1 t + \beta_1) + \alpha_2 b \cos(k_2 t + \beta_2) + c \sin t,$$

where, α, b, β_1, and β_2 are slowly varying functions of time t, and where

$$d = \frac{\lambda_1 Q}{1 + (\mp n_1^2 \mp n_2^2 - \lambda_1\lambda_2) + n_1^2 n_2^2},$$

$$c = \frac{-(1 \mp n_1^2)\, Q}{1 + (\mp n_1^2 \mp n_2^2 - \lambda_1\lambda_2) + n_1^2 n_2^2}$$

(the meaning and value of the other quantities is given in Section 10), then the approximate equations for a, b, β_1, and β_2 are

$$\frac{da}{d\tau} = -\frac{1}{n_1^3}\left(\lambda_1 A_3 - \frac{\lambda_2}{\alpha_1} A_1\right),$$

$$\frac{db}{d\tau} = \frac{1}{n_1^3}\left(\lambda_1 B_3 - \frac{\lambda_2}{\alpha_2} B_1\right),$$

$$\frac{d\beta_1}{d\tau} = -\frac{1}{a n_1^3}\left(\lambda_1 A_4 + \frac{\lambda_2}{\alpha_1} A_2\right),$$

$$\frac{d\beta_2}{d\tau} = \frac{1}{b n_1^3}\left(\lambda_1 B_4 + \frac{\lambda_2}{\alpha_2} B_2\right),$$

where

$$\tau = \frac{\mu n_1^3 t}{2(k_1^2 - k_2^2)}\,(k_1 > k_2),$$

$$A_1 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\xi \, d\xi \, d\eta \, dt,$$

$$A_2 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\xi \, d\xi \, d\eta \, dt,$$

$$A_3 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\xi \, d\xi \, d\eta \, dt,$$

$$A_4 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \cos\xi \, d\xi \, d\eta \, dt,$$

$$B_1 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \cos\eta \, d\xi \, d\eta \, dt,$$

$$B_2 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f^* \sin\eta \, d\xi \, d\eta \, dt,$$

$$B_3 = \frac{1}{4\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \sin\eta \, d\xi \, d\eta \, dt,$$

$$B_4 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} g^* \cos\eta \, d\xi \, d\eta \, dt,$$

$$f^* = f(a\sin\xi + b\sin\eta + d\cos t\ , ak_1\cos\xi + bk_2\cos\eta - d\sin t,$$
$$a\alpha_1\cos\xi + b\alpha_2\cos\eta + c\sin t, - a\alpha_1 k_1\sin\xi - b\alpha_2 k_2\sin\eta + c\cos t);$$
$$g^* = g(a\sin\xi + b\sin\eta + d\cos t, ak_1\cos\xi + bk_2\cos\eta - d\sin t,$$
$$a\alpha_1\cos\xi + b\alpha_2\cos\eta + c\sin t, - a\alpha_1 k_1\sin\xi - b\alpha_2 k_2\sin\eta + c\cos t);$$
$$\xi = k_1 t + \beta_1, \eta = k_2 t + \beta_2.$$

2. Let one of the natural frequencies be the same as the frequency of the external force.

Let us assume that $k_1^2 > k_2^2 = 1$, $Q = \mu Q'$, and look for a solution of the system (282) in the form

$$x = a \sin (k_1 t + \psi) + b_1 \sin t + b_2 \cos t,$$

$$y = \alpha_1 a \cos (k_1 t + \psi) + \alpha_2 b_1 \cos t - \alpha_2 b_2 \sin t,$$

where a, b_1, b_2, and ψ are slowly varying functions of time.

The equations for a, b_1, b_2, and ψ are

$$\frac{db_2}{dt} \cdot \frac{2(k_1^2 - 1)}{1 \mp n_2^2} = \mu\left(A_1 + \alpha_2 \frac{\lambda_1}{\lambda_2} B_1\right),$$

$$\frac{db_1}{dt} \cdot \frac{2(k_1^2 - 1)}{1 \mp n_2^2} = \mu\left[-A_2 + \alpha_2 \frac{\lambda_1}{\lambda_2}(B_2 + Q')\right],$$

$$\frac{da}{dt} \cdot \frac{2k_1(k_1^2 - 1)}{k_1^2 \mp n_2^2} = \mu\left(A_3 - \alpha_1 \frac{\lambda_1}{\lambda_2} B_3\right),$$

$$a\frac{d\psi}{dt} \cdot \frac{2k_1(k_1^2 - 1)}{k_1^2 \mp n_2^2} = -\mu\left(A_4 + \alpha_1 \frac{\lambda_1}{\lambda_2} B_4\right),$$

where

$$\left.\begin{aligned}
A_1 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \sin t \, d\xi \, dt, & B_1 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \cos t \, d\xi \, dt;\\
A_2 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \cos t \, d\xi \, dt, & B_2 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \sin t \, d\xi \, dt;\\
A_3 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \cos \xi \, d\xi \, dt, & B_3 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \sin \xi \, d\xi \, dt;\\
A_4 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^* \sin \xi \, d\xi \, dt, & B_4 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^* \cos \xi \, d\xi \, dt;
\end{aligned}\right\} \quad (283)$$

$$f^* = f(a \sin \xi + b_1 \sin t + b_2 \cos t, \, ak_1 \cos \xi + b_1 \cos t - b_2 \sin t,$$
$$\alpha_1 a \cos \xi + \alpha_2 b_1 \cos t - \alpha_2 b_2 \sin t, \, - \alpha_1 \sin \xi - \alpha_2 b_1 \sin t - \alpha_2 b_2 \cos t);$$

$$g^* = g(a \sin \xi + b_1 \sin t + b_2 \cos t, \, ak_1 \cos \xi + b_1 \cos t - b_2 \sin t,$$
$$\alpha_1 a \cos \xi + \alpha_2 b_1 \cos t - \alpha_2 b_2 \sin t, \, - \alpha_1 a k_1 \sin \xi - \alpha_2 b_1 \sin t - \alpha_2 b_2 \cos t);$$

$$\xi = k_2 t + \psi.$$

If $k_1^2 = 1$, then one can look for a solution in the form

$$x = a_1 \sin t + a_2 \cos t + b \sin (k_2 t + \psi),$$

$$y = \alpha_1 a_1 \cos t - \alpha_1 a_2 \sin t + \alpha_2 b \cos (k_2 t + \psi).$$

The equations for a_1, a_2, b, and ψ have the form:

$$\frac{da_2}{dt} \cdot \frac{2(1 - k_2^2)}{1 \mp n_2^2} = -\mu\left(A_1 + \alpha_1 \frac{\lambda_1}{\lambda_2} B_1\right),$$

$$\frac{da_1}{dt} \cdot \frac{2(1 - k_2^2)}{1 \mp n_2^2} = \mu\left[A_2 - \alpha_1 \frac{\lambda_1}{\lambda_2}(B_2 + Q')\right],$$

$$\frac{db}{dt} \cdot \frac{2k_2(1 - k_2^2)}{k_2^2 \mp n_2^2} = -\mu\left(A_3 - \alpha_2 \frac{\lambda_1}{\lambda_2} B_3\right),$$

$$b\frac{d\psi}{dt} \cdot \frac{2k_2(1 - k_2^2)}{k_2^2 \mp n_2^2} = \mu\left(A_4 + \alpha_2 \frac{\lambda_1}{\lambda_2} B_4\right),$$

where $A_{1,2,3,4}$ and $B_{1,2,3,4}$ are given by the formulas (283) in which

$$\xi = k_2 t + \psi;$$

$$f^* = f(a_1 \sin t + a_2 \cos t + b \sin \xi,\ a_1 \cos t - a_2 \sin t + bk_2 \cos \xi,$$
$$\alpha_1 a_1 \cos t - \alpha_1 a_2 \sin t + \alpha_2 b \cos \xi,\ -\alpha_1 a_1 \sin t - \alpha_1 a_2 \cos t - \alpha_2 k_2 b \sin \xi);$$

$$g^* = g(a_1 \sin t + a_2 \cos t + b \sin \xi,\ a_1 \cos t - a_2 \sin t + bk_2 \cos \xi,$$
$$\alpha_1 a_1 \cos t - \alpha_1 a_2 \sin t + \alpha_2 b \cos \xi,\ -\alpha_1 a_1 \sin t - \alpha_1 a_2 \cos t - \alpha_2 k_2 b \sin \xi).$$

The technique of using the derived equations for solving concrete problems is analogous to the technique used in Section 18, [15], [16].

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